Proof: We show the first assertion only because the second one can be shown analogously.

If $y_\alpha.v^+ = 0$, then $h_\alpha.v^+ = [x_\alpha, y_\alpha].v^+ = 0$. But $h_\alpha.v^+ = \lambda(h_\alpha)v^+ = \langle\lambda,\alpha\rangle v^+$, hence $\langle\lambda,\alpha\rangle = 0$. If $y_\alpha v^+ \neq 0$. Then $\lambda - \alpha \in \Pi(\lambda)$. On the other hand, $\lambda + \alpha \notin \Pi(\lambda)$ due to the maximality of λ, hence $\langle\lambda,\alpha\rangle \geq 1 > 0$ by (A.1). □

Theorem A10: *Let V and W be two irreducible L-modules. Let v^+ be a maximal vector of V and w^- a minimal of W. Then $v^+ \otimes w^-$ ($v^+ \wedge v^-$ resp. $v^+ \vee v^-$) generates the L-module $V \otimes W$ ($V \wedge V$ resp. $V \vee V$).*

Proof: Let U be the L-submodule of $V \otimes W$ generated by $v^+ \otimes w^-$. Apply y_{β_i} to $v^+ \otimes w^-$ repeatedly (notice that $y_\alpha.w^- = 0$), we get

$$y_{\beta_1}^{i_1} \cdots y_{\beta_m}^{i_m}.(v^+ \otimes w^-) = y_{\beta_1}^{i_1} \cdots y_{\beta_m}^{i_m}.v^+ \otimes w^-.$$

From Theorem A8 one concludes $V \otimes w^- \subseteq U$ because V is spanned by the elements $y_{\beta_1}^{i_1} \cdots y_{\beta_m}^{i_m}.v^+$.

Let W_0 be the collection of the elements $w \in W$ such that $V \otimes w \subseteq U$. W_0 is a subspace of W and $w^- \in W_0$. For any $x \in L$ and any $w \in W_0$, $V \otimes x.w \subseteq x.(V \otimes w) + x.V \otimes w \subseteq U$. This means W_0 is an invariant subspace of W. But W is irreducible, hence $W_0 = W$. □

Theorem A11: *(cf. [14] p. 109) Let L be a semisimple Lie algebra and V an irreducible L-module with maximal weight λ. If $L = L_1 \oplus L_2$ is the direct sum of semisimple Lie algebras L_i, then V can be identified with the tensor product $V_1 \otimes V_2$ of L_i-modules V_i with maximal weight $\lambda^{(i)} = \lambda|_{H_i}$, where $H_i := H \cap L_i$.*

From Theorem A11 we can conclude

Corollary A12: *The hypothesis is as in Theorem A11. If V is faithful in addition, then so are V_i ($i = 1, 2$); in particular, $\lambda^{(i)} \neq 0$.*

Proof: Let $x \in L_1$ and $x.V_1 = \{0\}$. Then $x.V = x.(V_1 \otimes V_2) = x.V_1 \otimes V_2 = \{0\}$. Because V is a faithful L-module, $x = 0$. □

For the element σ_0 for each type of simple Lie algebras one is referred to [14] p. 147-151. Recall that $\sigma \in W$ and $\sigma\Delta = -\Delta$.

Proposition A13: *Let L be a simple Lie algebra with the maximal root α_0. If $V(\lambda)$ is a faithful irreducible L-module with the property*

$$\alpha_0 + \alpha = \lambda - \sigma_0\lambda \tag{A.2}$$