$$\begin{aligned} &= -\alpha(\mathrm{ad}(v^+ \wedge v^-))x_\alpha \\ &\neq 0, \end{aligned}$$

which implies that $x_\alpha.v^- \neq 0$.

Now let $n > 3$. We proceed indirectly and suppose that for all $\alpha \in \Delta_{0,i}$: $x_\alpha.v^- = y_\alpha.v^+ = 0$. If $\Delta_{0,i} = \Delta_i$, then by the assumption $y_\alpha.v^+ = 0$ for all $\alpha \in \Delta_i$. Combining it with $x_\alpha.v^+ = 0$, we get $h_\alpha.v^+ = 0$, which implies that $\lambda(h_\alpha) = \langle\lambda, \alpha\rangle = 0$. So the restriction $\lambda^{(i)}$ of λ on H_i is zero, and we obtain a contradiction to the assumption that V is faithful L-module (see Corollary A12). Therefore $\Delta_{0,i}$ is a nonempty proper subset of Δ_i. Let $\alpha \in \Delta_{0,i}$ and $h = \mathrm{ad}(v^+ \wedge v^- \wedge v_1 \wedge \cdots \wedge v_{n-3}) \in H_0$ such that $\alpha(h) \neq 0$. Then since ad is a module morphism, we have

$$\alpha(h)y_\alpha = [y_\alpha, h] = \sum_{i=1}^{n-3} \mathrm{ad}(v^+ \wedge v^- \wedge v_1 \wedge \cdots \wedge y_\alpha.v_i \wedge \cdots \wedge v_{n-3}),$$

which in turn implies that $y_\alpha \in \mathrm{ad}(v^+ \wedge v^- \wedge V \wedge \cdots \wedge V)$. Analogously we can show that x_α is an element in $\mathrm{ad}(v^+ \wedge v^- \wedge V \wedge \cdots \wedge V)$, so is $h_\alpha = [x_\alpha, y_\alpha]$. Then $h_\alpha \in H_0$. Now for any $\beta \in \Delta_i \setminus \Delta_{0,i}$ and any $\alpha \in \Delta_{0,i}$: $\langle\beta, \alpha\rangle = \beta(h_\alpha) = 0$, that is, $\Delta_{0,i} \perp \Delta_i \setminus \Delta_{0,i}$, which contradicts that L_i is a simple ideal of L. Therefore the assumption for $\Delta_{0,i}$ is false.

Now we can prove Lemma 3.5 as follows.

Let $\alpha \in \Delta_{0,i}$ such that $x_\alpha.v^- \neq 0$ or $y_\alpha.v^+ \neq 0$. Since we can proceed analogously if $y_\alpha.v^+ \neq 0$, we assume that $x_\alpha.v^- \neq 0$. Further let $h \in H_0$, $h = \mathrm{ad}(v^+ \wedge v^- \wedge v_{\mu_1} \wedge \cdots \wedge v_{\mu_{n-3}})$ with $\alpha(h) \neq 0$. By pluging the expression for h in $[x_\alpha, h]$ we obtain:

$$\begin{aligned} &-\alpha(z)x_\alpha \\ &= \mathrm{ad}(v^+ \wedge x_\alpha.v^- \wedge v_{\mu_1} \wedge \cdots \wedge v_{\mu_{n-3}}) \\ &+ \sum_{j=1}^{n-3} \mathrm{ad}(v^+ \wedge v^- \wedge v_{\mu_1} \wedge \cdots \wedge v_{\mu_{j-1}} \wedge x_\alpha.v_{\mu_j} \wedge v_{\mu_{j+1}} \wedge \cdots \wedge v_{\mu_{n-3}}). \end{aligned}$$

If the element $\mathrm{ad}(v^+ \wedge v^- \wedge v_{\mu_1} \wedge \cdots \wedge x_\alpha.v_{\mu_j} \wedge \cdots \wedge v_{\mu_{n-3}})$ is nonzero for some j, it is a weight vector of weight α and we might assume that it agrees with x_α by choosing v^+ appropriately. Then we get $x_\alpha.v^- = \mathrm{ad}(v^+ \wedge v^- \wedge v_{\mu_1} \wedge \cdots \wedge x_\alpha.v_{\mu_i} \wedge \cdots \wedge v_{\mu_{n-3}}).v^- = 0$ which contradicts the assumption that $x_\alpha.v^- \neq 0$. Therefore all terms but the first one on the right side are 0, consequently $x_\alpha = \mathrm{ad}(v^+ \wedge x_\alpha.v^- \wedge v_{\mu_1} \wedge \cdots \wedge v_{\mu_{n-3}})$ (where z is choosen such that $\alpha(z) = -1$). From

$$\begin{aligned} 0 \neq x_\alpha.v^- &= \mathrm{ad}(v^+ \wedge x_\alpha.v^- \wedge v_{\mu_1} \wedge \cdots \wedge v_{\mu_{n-3}}).v^- \\ &= \mathrm{ad}(x_\alpha.v^- \wedge v^- \wedge v_{\mu_1} \wedge \cdots \wedge v_{\mu_{n-3}}).v^+ \end{aligned}$$