Appendix A

Lie algebras and their representations

We shall make some reviews and preparations in the theory of Lie algebras and their representations. We assume that all vector spaces, which will be considered, are finite dimensional over a field K of characteristic 0, unless otherwise is noted. For our notations and terminologies see Humphreys [7].

Recall that a Lie algebra is a vector space together with a bilinear operation $(x, y) \to [x, y]$ with the following properties:

1) $[x, y] = -[y, x]$,

2) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$

for all $x, y, z \in L$. The identity in 2) is the so-called Jacobi identity. A subspace I of L is called an ideal if $[I, L] \subseteq L$. If $[L, L] \neq \{0\}$ and if L and $\{0\}$ are the only ideals of L, then we say L is simple. L is called solvable if $L^{(s)} = \{0\}$ for some $s \in \mathbb{N}$, where $L^{(s)}$ is defined recursively via: $L^{(0)} := L$, $L^{(s+1)} := [L^{(s)}, L^{(s)}]$. The radical $Rad(L)$ is the unique maximal solvable ideal of L. By definition L is semisimple if $Rad(L) = \{0\}$; reductive if $Rad(L) = C(L)$, where $C(L)$ denotes the centre of L: $C(L) := \{x \in L \mid [x, L] = \{0\}\}$. It is obvious that a simple Lie algebra is semisimple, and therefore reductive.

Let V be a vector space. The set $gl(V)$ of all endomorphisms of V forms a Lie algebra relative to the Lie bracket $[f, g] := fg - gf$. A homomorphism ρ of a Lie algebra L into $gl(V)$ is said to be a representation of L in V. Define $x.v := \rho(x)v$, then we get a bilinear map $L \times V \longrightarrow V$ with $[x, y].v = x.y.v - y.x.v$ for all $x, y \in L$ and all $v \in V$. V together with such a map is called an L-module. Conversely if V is an L-module, then V defines a representation ρ of L by $\rho(x)(v) := x.v$. Hence we obtain two equivalent formulations. In this work we shall use both terminologies.

Given an L-module. V is called a faithful L-module if for all $x \in L$ $x.V = \{0\}$ implies $x = 0$. A submodule of V is a subspace U satisfying $L.U \subseteq U$. V is irreducible if $V \neq \{0\}$, and $\{0\}$ and V are the only submodules of V and completely reducible if each submodule possesses a complementary submodule. Further a linear map τ of V into an another L-module W is called an L-module morphism if $\tau(x.v) = x.\tau(v)$ for all $x \in L$ and $v \in V$.

In representation theory we have the well-known