L. We show

Lemma 3.7: *If L is a simple Lie algebra and (L, V, ad) is a good triple, then*

$$\lambda - \sigma_0\lambda = \alpha_0 + \alpha \tag{3.3}$$

for some simple root α of L.

Proof: To show the assertion we need the following representation for x_{α_0}:

$$x_{\alpha_0} = \mathrm{ad}(v_{\mu_1} \wedge \cdots \wedge v_{\mu_{n-2}} \wedge v^+) \tag{3.4}$$

for some $v_{\mu_i} \in V_{\mu_i}$, $\mu_i \in \Pi(\lambda)$, $i \in \underline{n-2}$. In fact, there exist $v_{\mu_i} \in V_{\mu_i}$, $\mu_i \in \Pi(\lambda)$, $i \in \underline{n-1}$, such that

$$x_{\alpha_0} = \mathrm{ad}(v_{\mu_1} \wedge \cdots \wedge v_{\mu_{n-1}}), \tag{3.5}$$

since ad is surjective and L_{α_0} is one dimensional. If $\mu_{n-1} \neq \lambda$, then there exist $w_{\nu_j} \in V_{\nu_j}$, $\nu_j \in \Pi(\lambda)$, and $\beta_j \in \Phi^+$, $j \in \underline{s}$ such that $v_{\mu_{n-1}} = \sum_{j=1}^{s} y_{\beta_j}.w_{\nu_j}$ (cf. Theorem A8). Inserting this expression for $v_{\mu_{n-1}}$ in (3.5), we get

$$\begin{aligned} x_{\alpha_0} &= \sum_{j=1}^{s}[y_{\beta_j}, \mathrm{ad}(v_{\mu_1} \wedge \cdots \wedge v_{\mu_{n-2}} \wedge w_{\nu_j})] \\ &\quad - \sum_{j=1}^{s}\sum_{i=1}^{n-2} \mathrm{ad}(v_{\mu_1} \wedge \cdots \wedge v_{\mu_{i-1}} \wedge y_{\beta_j}.v_{\mu_i} \wedge v_{\mu_{i+1}} \wedge \cdots \wedge v_{\mu_{n-2}} \wedge w_{\nu_j}). \end{aligned}$$

The first sum on the right side is zero because $\mathrm{ad}(v_{\mu_1} \wedge \cdots \wedge v_{\mu_{n-2}} \wedge w_{\nu_j}) \in L_{\alpha_0+\beta_j} = \{0\}$. Hence x_{α_0} is represented in the form as in (3.5), but the weight of the last component is greater. Repeating this process (we can do it only finitely many times) until the last position is occupied by a maximal vector, we have reached the expression (3.4).

We can show (3.3) by using (3.4) as follows. Since $\langle \sigma_0\lambda, \alpha_0\rangle = -\langle \lambda, \alpha_0\rangle \neq 0$ (otherwise $\langle \lambda, \alpha\rangle = 0$ for all $\alpha \in \Delta$, which implies that $\lambda = 0$ and V is one dimensional), we have $x_{\alpha_0}.v^- \neq 0$ by Proposition A9. On the other hand, we have by (3.4),

$$\mathrm{ad}(v_{\mu_1} \wedge \cdots \wedge v_{\mu_{n-2}} \wedge v^+).v^- = -\mathrm{ad}(v_{\mu_1} \wedge \cdots \wedge v_{\mu_{n-2}} \wedge v^-).v^+. \tag{3.6}$$

Therefore $\mathrm{ad}(v_{\mu_1} \wedge \cdots \wedge v_{\mu_{n-2}} \wedge v^-).v^+ \neq 0$, in particular, $\mathrm{ad}(v_{\mu_1} \wedge \cdots \wedge v_{\mu_{n-2}} \wedge v^-) \neq 0$. If its weight is $-\beta$, then $\beta \in \Phi^+ \cup \{0\}$ due to the maximality of v^+. By (3.6), $\alpha_0 + \sigma_0\lambda = \lambda - \beta$ or

$$\lambda - \sigma_0\lambda = \alpha_0 + \beta \tag{3.7}$$

If $\beta = 0$, then $\lambda - \sigma_0\lambda$ is the maximal root of L which is not true by Lemma 3.2. Therefore $\beta \in \Phi^+$. From Lemma 3.4 and (3.7) we get $\alpha_0 + \beta - \alpha \in \Phi$ for some simple