we have to calculate the real forms and the realification of all complex simple n-Lie algebras. As Theorem 3.8 shows, there is up to isomorphism only one finite dimensional complex simple n-Lie algebra $\widetilde{V}$ and this can be realized by means of a nondegenerate symmetric form b and a determinant form f ($\neq 0$) on a complex vector space $\widetilde{V}$ of dimension $n+1$ as in Example 1.1.1. Therefore as one simple real n-Lie algebra we have the realification of the vector product on $\mathbb{C}^{n+1}$ and it is of dimension $2(n+1)$ over $\mathbb{R}$.

Now we consider the real forms of $\widetilde{V}$. Since $\widetilde{V}$ is $(n+1)$-dimensional, each real form of $\widetilde{V}$ has also dimension $n+1$. By Lemma 1.2.1 any $(n+1)$-dimensional simple real n-Lie algebra V can be given as (K^{n+1}, b, f) as in Example 1.1.1. By Proposition 1.2.3 two such n-Lie algebras $(\mathbb{R}^{n+1}, b_1, f)$ and $(\mathbb{R}^{n+1}, b_2, f)$ are isomorphic if and only if there exists an automorphism τ of the space $\mathbb{R}^{n+1}$ with property (1.12). But each nondegenerate symmetric bilinear form b on $\mathbb{R}^{n+1}$ is congruent to one of the bilinear forms b_s, $0 \leq s \leq n+1$, whose associated matrix relative to the canonical base is $\mathrm{diag}(1, \cdots, 1, \underbrace{-1, \cdots -1}_{s})$, that is, there is an automorphism σ of $\mathbb{R}^{n+1}$ such that $b(\sigma u, \sigma v) = b_s(u, v)$. If $\det\sigma < 0$, we choose an element $\sigma' \in O(\mathbb{R}^{n+1}, b)$ with $\det\sigma' = -1$. Then $\det(\sigma'\sigma) > 0$ and $b(\sigma'\sigma u, \sigma'\sigma v) = b(\sigma u, \sigma v) = b_s(u, v)$. So we may assume that $\det\sigma > 0$. Set $\tau = (\det\sigma)^{-\frac{1}{n-1}}\sigma$. Then one can show that τ satisfies the identity (1.12) and is an isomorphism from $(\mathbb{R}^{n+1}, b, f)$ onto $(\mathbb{R}^{n+1}, b_s, f)$ (see Proposition 1.2.3). This means that every $(n+1)$-dimensional real simple n-Lie algebra is isomorphic to one of the n-Lie algebras $(\mathbb{R}^{n+1}, b_s, f)$, $0 \leq s \leq n+1$, where f is a fixed nonzero determinant form on $\mathbb{R}^{n+1}$ and b_s as above.

We claim that $(\mathbb{R}^{n+1}, b_s, f)$ is isomorphic to $(\mathbb{R}^{n+1}, b_{n+1-s}, f)$. In fact, let τ_1 be element in $O(\mathbb{R}^{n+1}, b_s)$ with $\det\tau_1 = -1$. Further let τ_2 be an isomorphism of $\mathbb{R}^{n+1}$ such that $-b_s(\tau_2 u, \tau_2 v) = b_{n+1-s}(u, v)$. As above we may assume that $\det\tau_2 > 0$. Set $\tau := (\det\tau_2)^{-\frac{1}{n-1}}\tau_1\tau_2$. Then we have $\det\tau = -(\det\tau_2)^{-\frac{2}{n-1}}$ and

$$\begin{aligned} b_s(\tau u, \tau v) &= (\det\tau_2)^{-\frac{2}{n-1}} b_s(\tau_1\tau_2 u, \tau_1\tau_2 v) \\ &= (\det\tau_2)^{-\frac{2}{n-1}} b_s(\tau_2 u, \tau_2 v) \\ &= -(\det\tau_2)^{-\frac{2}{n-1}} b_{n+1-s}(u, v) \\ &= \det\tau\ b_{n+1-s}(u, v). \end{aligned}$$

By Proposition 1.2.3, τ is an isomorphism from the n-Lie algebra $(\mathbb{R}^{n+1}, b_s, f)$ onto $(\mathbb{R}^{n+1}, b_{n+1-s}, f)$. Therefore we have proved that each real simple $n+1$-dimensional n-Lie algebra is isomorphic to one of the n-Lie algebras $(\mathbb{R}^{n+1}, b_s, f)$, $0 \leq s \leq [\frac{n+1}{2}]$.