$\lambda(h_\gamma)v^+ = \langle\lambda,\gamma\rangle v^+ \neq 0$. On the other hand, as ad is alternating, we have

$$\begin{aligned} 0 \neq x_\gamma.y_\gamma.v^+ &= \mathrm{ad}(v^+ \wedge v^- \wedge v_{\mu_1} \wedge \cdots \wedge v_{\mu_{n-3}}).y_\gamma.v^+ \\ &= \mathrm{ad}(y_\gamma.v^+ \wedge v^+ \wedge v_{\mu_1} \wedge \cdots \wedge v_{\mu_{n-3}}).v^-. \end{aligned}$$

In particular, $\mathrm{ad}(y_\gamma.v^+ \wedge v^+ \wedge v_{\mu_1} \wedge \cdots \wedge v_{\mu_{n-3}}) \neq 0$. By Lemma 3.1 the weight of this element is $2\lambda + \sum_{i=1}^{n-3}\mu_i - \gamma$, which is equal to $\lambda - \sigma_0\lambda$ by the definition of γ. Therefore $\lambda - \sigma_0\lambda \in \Phi \cup \{0\}$. By Lemma 3.2, $\lambda - \sigma_0\lambda \notin \Phi$, hence $\lambda - \sigma_0\lambda = 0$. Since $-\sigma_0\lambda$ is also a dominant weight, $\lambda = 0$, which is a contradiction to the assumption that V is faithful. □

<u>**Corollary 3.4**</u>: $H_0 \neq \{0\}$

Proof: If $n > 3$, the assertion follows from Lemma 3.3. Let $n = 3$. Since the nonzero element $\mathrm{ad}(v^+ \wedge v^-)$ lies in $L_{\lambda+\sigma_0\lambda}$ by Lemma 3.1, $\lambda + \sigma_0\lambda \in \Phi \cup \{0\}$. But $\sigma_0(\lambda+\sigma_0\lambda) = \sigma_0\lambda + \sigma_0^2\lambda = \lambda + \sigma_0\lambda$, hence $\lambda + \sigma_0\lambda = 0$ because σ_0 maps the positive roots into the negative ones. This means that $\mathrm{ad}(v^+ \wedge v^-) \in H_0$. □

We continue to show C2.

Let $L = \sum_{i=1}^m L_i$, where L_i, $i \in \underline{m}$ are the simple ideals of L. Further $H_i := H \cap L_i$, $\Phi_i \subseteq \Phi$ the root system of L_i relative to H_i and $\Delta_i := \Delta \cap \Phi_i$. Δ_i is a base of Φ_i (see Theorem A6 and Remark A1)

<u>**Lemma 3.5**</u>: *Let (L, V, ad) be a good triple. Then for each $i \in \underline{m}$ there exists $\alpha \in \Delta_i$ with $\lambda - \sigma_0\lambda - \alpha \in \Phi$.*

Proof: Set $\Delta_{0,i} := \{\alpha \in \Delta_i \mid \alpha(H_0) \neq \{0\}\}$. Then for each i: $\Delta_{0,i} \neq \phi$. If $\Delta_{0,i} = \phi$ for some $i \in \underline{m}$, say $i = 1$, then $H_0 \subseteq \oplus_{i=2}^m H_i$. Set $M := \oplus_{i=2}^m L_i$. We show that $\mathrm{ad}(V \wedge \cdots \wedge V) \subseteq M$, which contradicts the surjectivity of ad. Since $v^+ \wedge v^-$ generates $V \wedge V$, it suffices to show that

$$\mathrm{ad}(v^+ \wedge v^- \wedge V \wedge \cdots \wedge V) \subseteq M. \tag{3.2}$$

Let $z = \mathrm{ad}(v^+ \wedge v^- \wedge v_{\mu_1} \wedge \cdots \wedge v_{\mu_{n-3}}) \neq 0$. If $z \in H_0$, then $z \in M$; if $z \notin H_0$, then $z \in L_\gamma$ for some root γ. By Lemma 3.3, $\{0\} \neq [z, L_{-\gamma}] \subseteq H_0$, this in turn gives $z \in M$. Hence (3.2) is true and $\Delta_{0,i} \neq \phi$.

We claim that for each i there exists an $\alpha \in \Delta_{0,i}$ such that $x_\alpha.v^- \neq 0$ or $y_\alpha.v^+ \neq 0$.

If $n = 3$, then $\mathrm{ad}(v^+ \wedge v^-)$ is the only nonzero element in H_0 up to scalar. By the definition of $\Delta_{0,i}$ we have for all its elements α:

$$\mathrm{ad}(v^+ \wedge x_\alpha.v^-) = [x_\alpha, \mathrm{ad}(v^+ \wedge v^-)]$$