*Proof*: Since $\tau$ is an $L$-module morphism, it follows that for all $h \in H$:

$$\begin{aligned}
&[h, \tau(v_{\mu_1} \wedge \cdots \wedge v_{\mu_m})] \\
&= \sum_{i=1}^{m} \tau(v_{\mu_1} \wedge \cdots v_{\mu_{i-1}} \wedge h.v_{\mu_i} \wedge v_{\mu_{i+1}} \wedge \cdots \wedge v_{\mu_m}) \\
&= (\sum_{i=1}^{m} \mu_i)(h) \cdot \tau(v_{\mu_1} \wedge \cdots \wedge v_{\mu_m}).
\end{aligned}$$

Thus $\tau(v_{\mu_1} \wedge \cdots \wedge v_{\mu_m}) \in L_\gamma$ as asserted. □

Let $(L, V, \mathrm{ad})$ be a good triple and the notations for $L$ and $V$ be as above. Let $\sigma_0$ denote the element in the Weyl group $W$ of $L$ such that $\sigma_0\Delta = -\Delta$ (see Proposition A5). Then $\sigma_0\lambda$ is the minimal weight of $V$. Further let $v^+$ (resp. $v^-$) be a maximal (resp. minimal) weight vector of $V$. We determine $(L, V, \mathrm{ad})$ in three steps: We show

**C1)** *that $H$ contains a nonzero element of the form $ad(v^+ \wedge v^- \wedge v_1 \wedge \cdots \wedge v_{n-3})$, $v_i \in V$, $i \in \underline{n-3}$ if $n > 3$ and $ad(v^+ \wedge v^-)$ if $n = 3$ (see Corollary 3.4),*

**C2)** *that for each simple component of $L$ there exists a simple root $\alpha$ of this component such that $\lambda - \sigma_0\lambda - \alpha \in \Phi$ (see Lemma 3.5), and*

**C3)** *that the number of the irreducible representations with the property in C2) is finite. Then all good triples $(L, V, ad)$ can be found among them.*

We show the following lemma in advance.

**<u>Lemma 3.2</u>**: *If $(L, V, \mathrm{ad})$ is a good triple, then $\lambda - \sigma_0\lambda \notin \Phi$.*

*Proof*: Suppose that $\lambda - \sigma_0\lambda$ is a root of $L$. We will first show that this implies $L$ has to be simple. Suppose on the contrary that $L = L_1 \oplus L_2$, $L_i \neq \{0\}$, $i = 1, 2$. Let $H_i = H \cap L_i$. Then $H_i$ is a maximal toral subalgebra of $L_i$ (see Theorem A6). Let $\Phi_i \subseteq \Phi$ be the root system of $L_i$ relative to $H_i$. Let $\Delta_i := \Delta \cap \Phi_i$. Then $\Delta_i$ is a base of $\Phi_i$ (see Remark A1). Since $\Phi = \Phi_1 \cup \Phi_2$, $\lambda - \sigma_0\lambda$ is an element in $\Phi_1$ or in $\Phi_2$. Assume that $\lambda - \sigma_0\lambda \in \Phi_1$ (the other case can be treated analogously). Then $\lambda - \sigma_0\lambda$ vanishes on $H_2$. Let $\lambda^{(2)}$ be the restriction of $\lambda$ to $H_2$ and $\sigma_0 = \sigma_1 + \sigma_2$, where $\sigma_i$ is the element in the Weyl group of $L_i$ with $\sigma_i\Delta_i = -\Delta_i$. Then $\lambda - \sigma_0\lambda = (\lambda^{(1)} - \sigma_1\lambda^{(1)}) + (\lambda^{(2)} - \sigma_2\lambda^{(2)})$. So $(\lambda - \sigma_0\lambda)(H_2) = \{0\}$ can be translated into $\lambda^{(2)} - \sigma_2\lambda^{(2)} = \{0\}$. Since $\lambda^{(2)}$ and $\sigma_2\lambda^{(2)}$ are dominant weights of $L_2$, we get $\lambda^{(2)} = 0$, which implies that $L$ does not operate faithfully on $V$ (see Corollary A12). This contradicts our assumption that (L,V,ad) is a good triple and therefore $L$ is simple.