$$= \sum_{i=1}^{n}(-1)^{n-i}\mathrm{ad}_1(v_1,\cdots,\widehat{v_i},\cdots,v_n).\mathrm{ad}_1(u_1,\cdots,u_{n-2},v_i).w \tag{2.4}$$

(see (1.7)). Set $V := I + V_1$ (direct sum). For v_i from I or V ($i \in \underline{n}$) let

$$[v_1,\cdots,v_n] := \begin{cases} [v_1,\cdots,v_n]_1 & \text{if } v_i \in V_1 \text{ for all } i \\ (-1)^{n-i}\mathrm{ad}_1(v_1,\cdots,\widehat{v_i},\cdots,v_n).v_i & \text{if only } v_i \in I \\ 0 & \text{if at least two of } v_i \in I \end{cases}$$

Extending it linearly to V, then we get an alternating n-ary operation $[v_1,\cdots,v_n]$ on V with respect to which V becomes an n-Lie algebra.

In fact, for the $2n-1$ elements u_i ($i \in \underline{n-1}$) and v_i ($i \in \underline{n}$) in V_1 or I let

$$\begin{aligned} a &:= [u_1,\cdots,u_{n-1},[v_1,\cdots,v_n]] \\ b &:= \sum_{i=1}^{n}[v_1,\cdots,v_{i-1},[u_1,\cdots,u_{n-1},v_i],v_{i+1},\cdots,v_n]. \end{aligned}$$

If all u_i and all v_i are in V_1, then $a = b$, since V_1 is an n-Lie algebra; if only one of the u_i is in I and all v_i are in V_1 (since the product is alternating, we may assume that $u_{n-1} \in I$), then by (2.4)

$$\begin{aligned} a &= -[u_1,\cdots,u_{n-2},[v_1,\cdots,v_n],u_{n-1}] \\ &= -\mathrm{ad}_1(u_1,\cdots,u_{n-2},[v_1,\cdots,v_n]_1).u_{n-1} \\ &= -\sum_{i=1}^{n}(-1)^{n-i}\mathrm{ad}_1(v_1,\cdots,\widehat{v_i},\cdots,v_n).\mathrm{ad}_1(u_1,\cdots,u_{n-2},v_i).u_{n-1} \\ &= \sum_{i=1}^{n}(-1)^{n-i}[v_1,\cdots,\widehat{v_i},\cdots,v_n,[u_1,\cdots,u_{n-1},v_i]] \\ &= b; \end{aligned}$$

if all u_i are from V and only one of the v_i is from I, say v_n, then

$$\begin{aligned} a &= \mathrm{ad}_1(u_1,\cdots,u_{n-1}).\mathrm{ad}_1(v_1,\cdots,v_{n-1}).v_n \\ &= [\mathrm{ad}_1(u_1,\cdots,u_{n-1}),\mathrm{ad}_1(v_1,\cdots,v_{n-1})].v_n \\ &\quad +\mathrm{ad}_1(v_1,\cdots,v_{n-1}).\mathrm{ad}_1(u_1,\cdots,u_{n-1}).v_n \\ &= \sum_{i=1}^{n-1}\mathrm{ad}_1(v_1,\cdots,v_{i-1},[u_1,\cdots,u_{n-1},v_i]_1,v_{i+1},\cdots,v_{n-1}).v_n \\ &\quad +\mathrm{ad}(v_1,\cdots,v_{n-1}).\mathrm{ad}(u_1,\cdots,u_{n-1}).v_n \\ &= b; \end{aligned}$$