where U_i and U_i' are $\overline{L}_i$-modules. Then U_i is equivalent to U_i', in other words, $\wedge^{n_i} V_i$ and $\wedge^{n_i'} V_i$ contain equivalent submodules for all $i \in \underline{m}$. By Table 3, this is only possible when $n_i = n_i'$ or $n_i + n_i' = n + 1$ for all i. If $n_i = n_i'$ for all $i \in \underline{m}$, we are done. Assume that $n_i + n_i' = n + 1$ for some i, say $i = 1$. Then we have $\sum_{i=2}^m (n_i + n_i') = \sum_{i=2}^m n_i + \sum_{i=2}^m n_i' = 2n - (n_1 + n_1') = n - 1$, which forces $m \geq 2$ and $n_i + n_i' \leq n - 1 < n + 1$ for all $i \geq 2$. Note that for all $i \in \underline{m}$, one of the two possibilities $n_i = n_i'$, $n_i + n_i' = n + 1$ must hold. Therefore we get $n_i = n_i'$ for all $i \in \underline{m}$, $i \geq 2$ which in turn implies that $n_1 = n_1'$ because $\sum_{i=1}^m n_i = \sum_{i=1}^m n_i' = n$. The assertion follows.

8) *$[V_i, \cdots, V_i] \cong V_i$ as $\overline{L}$-modules for all $i \in \underline{m}$.* The $\overline{L}$-module $V_{(0,\cdots,n,\cdots,0)}$ is equivalent to a submodule of $V^{(0,\cdots,n,\cdots,0)}$ by 7) while $V^{(0,\cdots,n,\cdots,0)} = \wedge^n V_i \cong V_i$ because V_i is $(n+1)$-dimensional. Therefore $V_{(0,\cdots,n,\cdots,0)}$ is either $\{0\}$ or equivalent to V_i because V_i is irreducible. Once again by 7), the $\overline{L}$-module $V^{(n_1,\cdots,n_m)}$, $(n_1, \cdots, n_m) \neq (0, \cdots, n, \cdots, 0)$, contains no submodule equivalent to $V^{(0,\cdots,n,\cdots,0)}$, so does its submodule $V_{(n_1,\cdots,n_m)}$. Therefore the relation $V_{(0,\cdots,n,\cdots,0)} = \{0\}$ will imply that $[V_0, \cdots, V_0]$ contains no submodule equivalent to V_i which contradicts 6). Therefore $V_{(0,\cdots,n,\cdots,0)} = [V_i, \cdots, V_i] \cong V_i$.

9) *$V_{(n_1,\cdots,n_m)}$ is nonzero for at most one tuple $(n_1, \cdots, n_m)$ with $0 \leq n_1, \cdots, n_m < n$ and $n_1 + \cdots + n_m = n$, say $(n_1^0, \cdots, n_m^0)$.* Since I is an irreducible $\overline{L}$-module by 3), $V = I + \sum_{i=1}^m V_i$ being a direct sum of $m+1$ irreducible submodules. As a submodule of V, $[V_0, \cdots, V_0]$ will be a sum of at most $m+1$ irreducible submodules. By 8), $[V_i, \cdots, V_i] \cong V_i$ for all $i \in \underline{m}$, then the decomposition of $[V_0, \cdots, V_0]$ in 7) implies the assertion.

Now we can write

$$[V_0, \cdots, V_0] = \sum_{i=1}^m [V_i, \cdots, V_i] + V_{(n_1^0,\cdots,n_m^0)}.$$

In the following we discuss three cases.

Case 2.1: I is equivalent to no submodule of $[V_0, \cdots, V_0]$.

Because the decomposition of a module into its isotypic components is unique, I and V_i, $i \in \underline{m}$ are the only irreducible submodules of V (which are pairwise inequivalent). As a submodule of V, $[V_0, \cdots, V_0]$ must be a sum of some V_i's. Therefore V_0 is a (Levi) subalgebra of V.

Case 2.2: I is equivalent to a submodule of $V^{(n_1^0,\cdots,n_m^0)}$.

Once again I and V_i, $i \in \underline{m}$ are the only irreducible submodules of V and they are pairwise inequivalent. By 8), $[V_i, \cdots, V_i] = V_i$, that is, V_i is a subalgebra of V