Let τ be a vector space automorphism of K^{n+1}. Then τ is an isomorphism from V_1 onto V_2 if and only if

$$b_2(\tau[v_1,\cdots,v_n]_1,\tau v_{n+1}) = b_2([\tau v_1,\cdots,\tau v_n]_2,\tau v_{n+1}) \tag{1.11}$$

for all $v_i \in V_1$, $i \in \underline{n+1}$. For the right side of the identity we have

$$\begin{aligned} b_2([\tau v_1,\cdots,\tau v_n]_2,\tau v_{n+1}) &= f(\tau v_1,\cdots,\tau v_{n+1}) \\ &= \det\tau \cdot f(v_1,\cdots,v_{n+1}) \\ &= \det\tau \cdot b_1([v_1,\cdots,v_n]_1,v_{n+1}). \end{aligned}$$

Thus (1.11) can be written as

$$b_2(\tau[v_1,\cdots,v_n]_1,\tau v_{n+1}) = \det\tau \cdot b_1([v_1,\cdots,v_n]_1,v_{n+1}).$$

Since V_1 is simple by Proposition 1.1.2, $[V_1,\cdots,V_1]_1 = V_1$. Thus τ is an isomorphism from V_1 onto V_2 if and only if τ satisfies the identity

$$b_2(\tau u,\tau v) = \det\tau \cdot b_1(u,v) \tag{1.12}$$

for all $u,v \in K^{n+1}$. Since for each bilinear form b there exists a base of K^{n+1} relative to which the associated matrix of b has diagonal form and each diagonal matrix multiplied by a scalar remains diagonal, any $(n+1)$-dimensional simple n-Lie algebra is isomorphic to one of the n-Lie algebras (K^{n+1},b,f), where b runs through the set of the bilinear forms whose associated matrix relative to the canonical base is $\mathrm{diag}(\alpha_1,\alpha_2,\cdots,\alpha_{n+1})$, where $\alpha_i \in K$, $i \in \underline{n+1}$ and $\alpha_1\cdots\alpha_{n+1} \neq 0$.

If K is algebraically closed, there exists a vector space automorphism σ of K^{n+1} such that $b_2(\sigma u,\sigma v) = b_1(u,v)$. Set $\tau := (\det\sigma)^{-\frac{1}{n-1}}\sigma$. Then τ is clearly also an automorphism of K^{n+1} and fulfills identity (1.12). Indeed, because of $\det\tau = (\det\sigma)^{-\frac{n+1}{n-1}}\det\sigma = (\det\sigma)^{-\frac{2}{n-1}}$ we have

$$\begin{aligned} b_2(\tau u,\tau v) &= (\det\sigma)^{-\frac{2}{n-1}} \cdot b_2(\sigma u,\sigma v) \\ &= (\det\sigma)^{-\frac{2}{n-1}} \cdot b_1(u,v) \\ &= \det\tau \cdot b_1(u,v). \end{aligned}$$

Hence V_1 is isomorphic to V_2 and we have shown

Proposition 1.2.3: *(K^{n+1},b_1,f) is isomorphic to (K^{n+1},b_2,f) if and only if there exists an isomorphism τ of K^{n+1} with property (1.12). If K is algebraically closed, then all n-Lie algebras of the form (K^{n+1},b,f) are isomorphic to each other.*