$I \subseteq V$ is a $Der(V)$- or an $Inder(V)$-submodule, then $\mathrm{ad}(u_1, \cdots, u_{n-1})I \subseteq I$ for all $u_1, \cdots, u_{n-1} \in V$, therefore I is an ideal of V.

We close this section with the following theorem.

Theorem 1.1.3: *Let V a simple n-Lie algebra over K, then $Der(V)$ and $Inder(V)$ act irreducibly on V. If K is of characteristic 0 and V is finite dimensional in addition, then*

1) all derivations of V are inner,

2) $Der(V)$ is semisimple.

For its proof we need

Lemma 1.1.4: *Let V be an n-Lie algebra over K, $\mathrm{char}K = 0$, with $C(V) = \{0\}$ or $[V, \cdots, V] = V$. If D is a derivation of V which commutes with all inner derivations of V, then $D = 0$.*

Proof: Let $v_1, \cdots, v_n \in V$. For each $i \in \underline{n}$ we have by the assumption,

$$\begin{aligned} D[v_1, \cdots, v_n] &= (-1)^{n-i} D\, \mathrm{ad}(v_1, \cdots, \widehat{v_i}, \cdots, v_n)v_i \\ &= (-1)^{n-i}\mathrm{ad}(v_1, \cdots, \widehat{v_i}, \cdots, v_n)Dv_i \\ &= [v_1, \cdots, v_{i-1}, Dv_i, v_{i+1}, \cdots, v_n], \end{aligned}$$

that is, $D[v_1, \cdots, v_n] = [v_1, \cdots, v_{i-1}, Dv_i, v_{i+1}, \cdots, v_n]$. From (1.9) we get

$$D[v_1, \cdots, v_n] = n\, D[v_1, \cdots, v_n].$$

Since $n \geq 2$, $D[v_1, \cdots, v_n] = 0$ for all $v_i \in V$, $i \in \underline{n}$. If $[V, \cdots, V] = V$, then $Dv = 0$ for all $v \in V$ or $D = 0$. Since $[Dv_1, \cdots, v_n] = D[v_1, v_2, \cdots, v_n] = 0$, $D(V) \subseteq C(V)$. Therefore $D = 0$ if $C(V) = \{0\}$. □

Proof of Theorem 1.1.3:

Set $L' := Der(V)$, $L := Inder(V)$. The first assertion is obvious, since an L'- or L-submodule of V is also an ideal of the n-Lie algebra V. Now let K be of characteristic 0. Since L' operates faithfully and irreducibly on V, it is a reductive Lie algebra (see Theorem A2) , that is, L can be represented as the direct sum of its centre Z and the semisimple Lie algebra $[L', L']$. But each element in Z commutes with those of L and $C(V) = \{0\}$, hence $Z = \{0\}$ by Lemma 1.1.4. Thus L' is semisimple.

Let L_1 be the ideal of L' such that $L' = L \oplus L_1$. Since $[L_1, L] = \{0\}$, it follows once again from Lemma 1.1.4 that $L_1 = \{0\}$. Thus $L' = L$. □