1.2 Simple n-Lie Algebras of Dimension $n+1$

We are concerned here with $(n+1)$-dimensional simple n-Lie algebras. It turns out that each $(n+1)$-dimensional simple n-Lie algebra can be realized as (K^{n+1}, b, f) for some nondegenerate symmetric bilinear form b and some nonzero determinant form f.

Let V be a simple n-Lie algebra over K of dimension $n+1$ with product $[v_1, \cdots, v_n]$. Since $[V, \cdots, V] = V$, the n-Lie product gives an isomorphism $\tau : \wedge^n V \to V$ with $\tau(v_1 \wedge \cdots \wedge v_n) = [v_1, \cdots, v_n]$. Let f be an arbitrary nonzero determinant form on V. We define an isomorphism $\sigma : \wedge^n V \to V^*$ by $\sigma(v_1 \wedge \cdots \wedge v_n)(v_{n+1}) = f(v_1, \cdots, v_{n+1})$, for $v_i \in V$, $i \in \underline{n+1}$. For $u, v \in V$, let $b(u,v) := (\sigma\tau^{-1}(u))(v)$. The bilinear form b has the following properties:

1) b is nondegenerate,

2) for all $v_i \in V$, $i \in \underline{n+1} : b([v_1, \cdots, v_n], v_{n+1}) = f(v_1, \cdots, v_{n+1})$,

3) b is symmetric.

Proof. 1) If $b(u,v) = 0$ for all $v \in V$, then $\sigma\tau^{-1}(u) = 0$ which implies $u = 0$, since σ and τ are isomorphisms. Hence b is nondegenerate.

2) For all $v_i \in V$, $i \in \underline{n+1}$:

$$\begin{aligned} b([v_1, \cdots, v_n], v_{n+1}) &= (\sigma\tau^{-1}([v_1, \cdots v_n]))(v_{n+1}) \\ &= \sigma(v_1 \wedge \cdots \wedge v_n)(v_{n+1}) \\ &= f(v_1, \cdots, v_{n+1}). \end{aligned}$$

3) Let u_i, $v_i \in V$, $i \in \underline{n}$. By (1.3), the G.J.I and 2) we get

$$\begin{aligned} & b([u_1, \cdots, u_n], [v_1, \cdots, v_n]) \\ &= f(u_1, \cdots, u_n, [v_1, \cdots, v_n]) \\ &= \mathrm{tr}(\mathrm{ad}(v_1, \cdots, v_{n-1}))\, f(u_1, \cdots, u_n, v_n) \\ &\quad - \sum_{i=1}^{n} f(u_1, \cdots, u_{i-1}, [v_1, \cdots, v_{n-1}, u_i], u_{i+1}, \cdots, u_n, v_n) \\ &= \mathrm{tr}(\mathrm{ad}(v_1, \cdots, v_{n-1}))\, f(u_1, \cdots, u_n, v_n) \\ &\quad - \sum_{i=1}^{n} b([u_1, \cdots, u_{i-1}, [v_1, \cdots, v_{n-1}, u_i], u_{i+1}, \cdots, u_n], v_n) \\ &= \mathrm{tr}(\mathrm{ad}(v_1, \cdots, v_{n-1}))\, f(u_1, \cdots, u_n, v_n) \\ &\quad - b([v_1, \cdots, v_{n-1}, [u_1, \cdots, u_n]], v_n) \\ &= \mathrm{tr}(\mathrm{ad}(v_1, \cdots, v_{n-1}))\, f(u_1, \cdots, u_n, v_n) - \end{aligned}$$