$$\begin{aligned}
&f(v_1,\cdots,v_{n-1},[u_1,\cdots,u_n],v_n)\\
&=\mathrm{tr}(\mathrm{ad}(v_1,\cdots,v_{n-1}))\,f(u_1,\cdots,u_n,v_n)\\
&\quad+f(v_1,\cdots,v_n,[u_1,\cdots,u_n])\\
&=\mathrm{tr}(\mathrm{ad}(v_1,\cdots,v_{n-1}))\,f(u_1,\cdots,u_n,v_n)\\
&\quad+b([v_1,\cdots,v_n],[u_1,\cdots,u_n]).
\end{aligned}$$

Therefore

$$\begin{aligned}
&b([u_1,\cdots,u_n],[v_1,\cdots,v_n])-b([v_1,\cdots,v_n],[u_1,\cdots,u_n])\\
&=\mathrm{tr}(\mathrm{ad}(v_1,\cdots,v_{n-1}))f(u_1,\cdots,u_n,v_n).
\end{aligned}\tag{1.10}$$

Suppose that $v_{n-1}\neq 0$. Let $v_n:=v_{n-1}$ and let $\{u_1,\cdots,u_n,v_n\}$ be a base of V. Then we get from (1.10) that $\mathrm{tr}(\mathrm{ad}(v_1,\cdots,v_{n-1}))\,f(u_1,\cdots,u_n,v_n)=0$, which leads to $\mathrm{tr}(\mathrm{ad}(v_1,\cdots,v_{n-1}))=0$. Now (1.10) becomes

$$b([u_1,\cdots,u_n],[v_1,\cdots,v_n])=b([v_1,\cdots,v_n],[u_1,\cdots,v_n]).$$

Since V is simple, $[V,\cdots,V]=V$, the symmetry of b follows. □

Proposition 1.2.1: *Let V be an $(n+1)$-dimensional n-Lie algebra. If V is simple, then $V=(K^{n+1},b,f)$ for some nondegenerate symmetric bilinear form b and a nonzero determinant form f.*

The converse is also true.

Proposition 1.2.2: *The n-Lie algebra (K^{n+1},b,f) is simple.*

Proof: Set $V:=(K^{n+1},b,f)$. Let I be a nonzero proper ideal of V. Set $I^\perp:=\{v\in V\mid b(v,u)=0,\forall u\in I\}$. Then $I^\perp\neq\{0\}$. Let $v_n\in I$ and $v_{n+1}\in I^\perp$ be nonzero elements. If v_n and v_{n+1} are not proportional to each other, then we can choose $v_1,\cdots,v_{n-1}\in V$ so, that $\{v_1,\cdots,v_{n+1}\}$ is a base of V. Since $[v_1,\cdots,v_n]\in I$, we have $f(v_1,\cdots,v_{n+1})=b([v_1,\cdots,v_n],v_{n+1})=0$, which is a contradiction to $f\neq 0$. Thus $I=I^\perp$ and is one dimensional. It follows that $\dim(K^{n+1})=\dim(I)+\dim(I^\perp)=2$ and $n=1$. Hence V possesses no nonzero proper ideal, i.e. V is simple. □

In the following we study when two n-Lie algebras of the form (K^{n+1},b,f) are isomorphic. Since the n-Lie algebra $(K^{n+1},b,\alpha f)$ is the same as the n-Lie algebra $(K^{n+1},\alpha^{-1}b,f)$ for all $\alpha\in K$, $\alpha\neq 0$, we can fix f and assume that $f(e_1,\cdots,e_{n+1})=1$, where $\{e_1,\cdots,e_{n+1}\}$ is the canonical base of K^{n+1}.

Let $V_i:=(K^{n+1},b_i,f)$, $i=1,2$, be two n-Lie algebras defined as in Example 1.1.1. The n-Lie product $[v_1,\cdots,v_n]_i$ of V_i satisfies the identity:

$$b_i([v_1,\cdots,v_n]_i,v_{n+1})=f(v_1,\cdots,v_{n+1}).$$