Chapter 3

Classification of simple n-Lie algebras

Let K be an algebraically closed field of characteristic 0. Theorem 2.7 shows that the finite dimensional semisimple n-Lie algebras over K can be decomposed into a direct sum of simple n-Lie algebras. Therefore in order to determine the finite dimensional semisimple n-Lie algebras we have to study the finite dimensional simple n-Lie algebras over K. By Theorem 1.1.3 the derivation algebra of such an n-Lie algebra is semisimple and acts irreducibly on the n-Lie algebra itself. This suggests the theory of representations of semisimple Lie algebras for our purpose. In this chapter we shall show that there is for every $n \geq 3$ only one finite dimensional simple n-Lie algebra over K up to isomorphism and this is just the n-Lie algebra with the vector product. With the help of this result we shall give all finite dimensional real simple n-Lie algebras up to isomorphism.

First let K be an arbitrary field. Let V be an n-Lie algebra over K. Notice that $\mathrm{ad} : \times^{n-1}V \to Inder(V)$ is a map such that for all $D \in Inder(V)$,

$$[D,\ \mathrm{ad}(v_1,\cdots,v_{n-1})] = \sum_{i=1}^{n-1} \mathrm{ad}(v_1,\cdots,v_{i-1},Dv_i,v_{i+1},\cdots,v_{n-1}),$$

and the the associated map $(v_1,\cdots,v_n) \to \mathrm{ad}(v_1,\cdots,v_{n-1})v_n$ from $\times^n V$ to V is alternating. If we regard V as an $Inder(V)$-module, then ad induces an $Inder(V)$-module morphism from $\wedge^{n-1}V$ to $Inder(V)$ (which we denote also by ad) such that the map $(v_1,\cdots,v_n) \to \mathrm{ad}(v_1 \wedge \cdots \wedge v_{n-1})v_n$ is alternating. Conversely, if (L, V, ad) is a triple with L a Lie algebra, V an L-module and ad an L-module morphism from $\wedge^{n-1}V$ to L such that the map $(v_1,\cdots,v_n) \to \mathrm{ad}(v_1 \wedge \cdots \wedge v_{n-1})v_n$ from $\times^n V$ to V is alternating, then V becomes an n-Lie algebra by

$$[v_1,\cdots,v_n] := \mathrm{ad}(v_1 \wedge \cdots \wedge v_{n-1}).v_n.$$

Therefore we get a correspondence between the set of n-Lie algebras and the set of the triples (L, V, ad).

Let $\tau : V_1 \to V_2$ be an n-Lie algebra isomorphism. Let $L_i := Inder(V_i)$ and let ad_i be the map from $\wedge^{n-1}V_i \to L_i$, $i = 1, 2$. Then $\gamma : L_1 \to L_2$, defined by $\gamma(D)(\tau v) = \tau(Dv)$, is a Lie algebra isomorphism (see (2.1)) and

$$\gamma(\mathrm{ad}_1(v_1 \wedge \cdots \wedge v_{n-1})) = \mathrm{ad}_2(\tau v_1 \wedge \cdots \wedge \tau v_{n-1}) \tag{3.1}$$