2) $V = C(V) \oplus V_0$, where $C(V)$ is the centre of V and V_0 is a semisimple n-Lie subalgebra of V.

3) V is reductive.

Proof: 1) $\Rightarrow$ 2) If L $(= Inder(V))$ is semisimple, then the L-module V is completely reducible. We decompose V into a direct sum of irreducible submodules: $V = \oplus V_i$. This is at the same time a direct sum of ideals of V. If V_i is one dimensional, that is, $L(V_i) = \{0\}$, then $V_i \subseteq C(V)$. If V_i is not trivial, then V_i is simple as a subalgebra of V (see the proof of Theorem 2.7). Hence $V = C(V) \oplus V_0$, where V_0 is the sum of the V_i with $\dim(V_i) > 1$ and thus semisimple.

2) $\Rightarrow$ 3) It is clear by Theorem 2.5.

3) $\Rightarrow$ 1) Suppose that V is reductive, that is, $Rad(V)$ is the centre $C(V)$ of V. Let R denote the radical of L. Since R is included in the radical of $Der(V)$ (note R is included in the intersectionof $Inder(V)$ with the radical of $Der(V)$, cf. [18] p. 204), we have $R(V) \subseteq C(V)$ by Theorem 2.9. Now

$$[R, \mathrm{ad}(V, \cdots, V)] \subseteq \mathrm{ad}(C(V), V, \cdots, V) = \{0\}$$

implies $R \subseteq Z$, the centre of L, thus L is reductive. Let L_0 be the semisimple subalgebra with $L = Z \oplus L_0$.

It remains to show that $Z = \{0\}$. Let us regard V as an L_0-module. Since L_0 is semisimple, there exists an L_0-invariant subspace V_0 of V such that $V = C(V) \oplus V_0$. Since $Z(C(V)) = \{0\}$, it suffices to prove that $Z(V_0) = \{0\}$. Let π denote the canonical homomorphism from V onto $V/Rad(V)$ and let γ be as in Theorem 2.9. Then $\gamma(L_0) = \gamma(L) = Inder(V/Rad(V))$ and $\pi(V_0) = \pi(V) = V/Rad(V)$. It follows that $\pi(L_0(V_0)) = \gamma(L_0)(\pi(V_0)) = V/Rad(V)$. Comparing dimension shows that $L_0(V_0) = V_0$. From $[Z, L_0] = \{0\}$ and $Z(V) \subseteq C(V)$ we get: $Z(V_0) = Z(L_0(V_0)) = L_0(Z(V_0)) \subseteq L_0(C(V)) = \{0\}$. □

One might ask whether every n-Lie algebra V possesses an ideal V_0 such that V is the direct sum of V_0 and its radical. The answer is no. In the following example we construct an n-Lie algebra $V = I + V_1$, where I is an abelian ideal and V a subalgebra (not an ideal) of V and $I \cap V_1 = \{0\}$. If V_1 is semisimple, then I is the radical of V. We shall see in chapter 5 that each n-Lie algebra over an algebraically closed field of characteristic 0 is a vector space direct sum of its radical and a semisimple subalgebra.

Example 2.4: Let V_1 be an n-Lie algebra with product $[v_1, \cdots, v_n]_1$. Let I be an $Inder(V_1)$-module such that

$$\mathrm{ad}_1(u_1, \cdots, u_{n-2}, [v_1, \cdots, v_n]_1).w$$