Chapter 1

n-Lie algebras

Throughout this chapter K will be a field of characteristic different from 2 except where otherwise noted.

1.1 Definitions and Examples

In this section we shall introduce some basic concepts for n-Lie algebras and discuss some examples.

Let us begin with the well-known Lie algebra $\mathbb{R}^3$ with the vector product $[x, y]$ which can be realized by means of a scalarproduct b and a determinant form f in the following way: $b([x, y], z) = f(x, y, z), \quad x, y, z \in \mathbb{R}^3$. This Lie algebra is isomorphic to $su(2)$ whose complexification is isomorphic to $sl(2, \mathbb{C})$ which plays an important role in the theory of finite dimensional complex semisimple Lie algebras. We generalize the vector product structure to K^n, $n \geq 3$, in the following way.

<u>Example 1.1.1</u>: Let $n \in \mathbb{N}$, $n \geq 2$. Let f be a nonzero determinant form and b a nondegenerate symmetric bilinear form on K^{n+1}. Given any fixed $v_i \in K^{n+1}$, $i \in \underline{n}$, there exists a unique element $[v_1, \cdots, v_n] \in K^{n+1}$ depending only on $v_1, \cdots, v_n$ such that for all $x \in K^{n+1}$: $f(v_1, \cdots, v_n, x) = b([v_1, \cdots, v_n], x)$. Obviously the map $(v_1, \cdots, v_n) \to [v_1, \cdots, v_n]$ is n-linear and alternating. Moreover we find the following property which one might think of as a generalization of the Jacobi identity:

$$[u_1, \cdots, u_{n-1}, [v_1, \cdots, v_n]] = \sum_{i=1}^{n} [v_1, \cdots, v_{i-1}, [u_1, \cdots, u_{n-1}, v_i], v_{i+1}, \cdots, v_n], \quad (1.1)$$

where $u_i \in K^{n+1}$, $i \in \underline{n-1}$ and $v_j \in K^{n+1}$, $j \in \underline{n}$.

In order to verify this identity let us define for fixed $u_i \in K^{n+1}$, $i \in \underline{n-1}$, an endomorphism $\mathrm{ad}(u_1, \cdots, u_{n-1})$ of K^{n+1} via $\mathrm{ad}(u_1, \cdots, u_n)v := [u_1, \cdots, u_n, v]$, $v \in K^{n+1}$. Then (1.1) can be written equivalently as

$$\mathrm{ad}(u_1, \cdots, u_{n-1})[v_1, \cdots, v_n] = \sum_{i=1}^{n} [v_1, \cdots, \mathrm{ad}(u_1, \cdots, u_{n-1})v_i, \cdots, v_n]. \quad (1.2)$$