1) the map is n-linear,

2) the map is alternating,

3) identity (1.1) holds for all u_i, $v_j \in V$, $i \in \underline{n-1}$, $j \in \underline{n}$.

We call identity (1.1) the generalized Jacobi identity (G.J.I.) and the endomorphisms $\mathrm{ad}(u_1, \cdots, u_{n-1})$, defined by $\mathrm{ad}(u_1, \cdots, u_{n-1})v := [u_1, \cdots, u_{n-1}, v]$, left multiplications. With the help of left multiplications the G.J.I. can also be rewritten as

$$\begin{aligned}&[\mathrm{ad}(u_1, \cdots, u_{n-1}), \mathrm{ad}(v_1, \cdots, v_{n-1})]\\ &= \sum_{i=1}^{n-1} \mathrm{ad}(v_1, \cdots, v_{i-1}, \mathrm{ad}(u_1, \cdots, u_{n-1})v_i, v_{i+1}, \cdots, v_{n-1}),\end{aligned} \tag{1.6}$$

that is, the Lie product of two left multiplications is the sum of some left multiplications. One can prove that (1.1) is also equivalent to

$$\begin{aligned}&\mathrm{ad}(u_1, \cdots, u_{n-2}, [v_1, \cdots, v_n])\\ &= \sum_{i=1}^{n} (-1)^{n-i} \mathrm{ad}(v_1, \cdots, \widehat{v_i}, \cdots, v_n) \mathrm{ad}(u_1, \cdots, u_{n-2}, v_i).\end{aligned} \tag{1.7}$$

Therefore we get four equivalent identities (1.1), (1.2), (1.6) and (1.7).

We remark that in case $n = 2$ this definition agrees with that of a Lie algebra. Therefore the concept of n-Lie algebras generalizes that of Lie algebras.

Example 1.1 shows that K^{n+1} carries the structure of an n-Lie algebra with respect to the product defined by means of a nondegenerate symmetric bilinear form b and a nonzero determinant form f on V. This n-Lie algebra will be denoted by (K^{n+1}, b, f).

Let us look at some more examples.

Example 1.1.2: Let A be an associative commutative algebra over K. Let D_i, $i \in \underline{n}$, be derivations of A satisfying $D_i D_j = D_j D_i$, for all i, $j \in \underline{n}$. We put for $a_i \in A$, $i \in \underline{n}$,

$$[a_1, \cdots, a_n] := \det(D_i a_j)_{(i,j) \in \underline{n} \times \underline{n}}.$$

Then the vector space A equipped with this product becomes an n-Lie algebra.

Proof: Suppose that D is a derivation of A of the form $D = \sum_{i=1}^{n} x_i D_i$, then $[D_j, D] = D_j D - D D_j = \sum_{i=1}^{n} (D_j x_i) D_i$. For this D we have the identity

$$D[a_1, \cdots, a_n] - \sum_{j=1}^{n} [a_1, \cdots, D a_j, \cdots, a_n] = -\left(\sum_{i=1}^{n} D_i x_i\right) \cdot [a_1, \cdots, a_n]. \tag{1.8}$$