To prove (1.2) we need a simple fact: Let T be an endomorphism of K^{n+1}, then for all $v_i \in K^{n+1}$, $i \in \underline{n+1}$:

$$\sum_{i=1}^{n+1} f(v_1, \cdots, v_{i-1}, Tv_i, v_{i+1}, \cdots, v_{n+1}) = \operatorname{tr}(T) \cdot f(v_1, \cdots, v_{n+1}). \tag{1.3}$$

By definition of $[v_1, \cdots, v_n]$, identity (1.3) is equivalent to

$$\begin{aligned} &\sum_{i=1}^{n} b([v_1, \cdots, v_{i-1}, Tv_i, v_{i+1}, \cdots, v_n], v_{n+1}) \\ &+b([v_1, \cdots, v_n], Tv_{n+1}) = \operatorname{tr}(T) \cdot b([v_1, \cdots, v_n], v_{n+1}). \end{aligned} \tag{1.4}$$

If $T \in so(K^{n+1}, b)$, that is, T is an endomorphism of K^{n+1} such that $b(Tu, v) + b(u, Tv) = 0$, then (1.4) becomes

$$\sum_{i=1}^{n} b([v_1, \cdots, v_{i-1}, Tv_i, v_{i+1}, \cdots, v_n], v_{n+1}) = b(T[v_1, \cdots, v_n], v_{n+1})$$

because of $\operatorname{tr}(T) = 0$. Since v_{n+1} is arbitrary in K^{n+1} and b is nondegenerate, we conclude that for all $T \in so(K^{n+1}, b)$ equation (1.4) is equivalent to

$$T[v_1, \cdots, v_n] = \sum_{i=1}^{n} [v_1, \cdots, v_{i-1}, Tv_i, v_{i+1}, \cdots, v_n]. \tag{1.5}$$

Thanks to (1.5) it remains to show that $\operatorname{ad}(u_1, \cdots, u_n) \in so(K^{n+1}, b)$ in order to get (1.2). Indeed, since b is symmetric and f is alternating, we have

$$\begin{aligned} b(\operatorname{ad}(u_1, \cdots, u_{n-1})v, w) &= b([u_1, \cdots, u_{n-1}, v], w) \\ &= f(u_1, \cdots, u_{n-1}, v, w) \\ &= -f(u_1, \cdots, u_{n-1}, w, v) \\ &= -b([u_1, \cdots, u_{n-1}, w], v) \\ &= -b(\operatorname{ad}(u_1, \cdots, u_{n-1})w, v) \\ &= -b(v, \operatorname{ad}(u_1, \cdots, u_{n-1})w). \end{aligned}$$

In this work we shall study the algebraic structure which arises in the above example. We describe this kind of structure abstractly in a few axioms.

Definition of n-Lie algebra:

Let $n \in \mathbb{N}$, $n \geq 2$. A vector space V over K together with a map $(v_1, \cdots, v_n) \to [v_1, \cdots, v_n]$ of $\times^n V$ into V is called an n-Lie algebra if the following properties are satisfied: