Chapter 4

# Levi decomposition

Throughout this chapter $K$ will be an algebraically closed field of characteristic 0 and all vector spaces over $K$ will be finite dimensional.

We know that any Lie algebra $L$ over $K$ admits a Levi decomposition, that is, there exists a (Levi) subalgebra $L_0$ of $L$ such that $L = Rad(L) + L_0$ and $L_0 \cap Rad(L) = \{0\}$ (cf. [18] p. 225). In this chapter we shall show using this results for Lie algebras that $n$-Lie algebras ($n \geq 3$) over $K$ have an analogous decomposition. Given an $n$-Lie algebra $V$, we call a subalgebra $V_0$ a Levi subalgebra of $V$ if $V = Rad(V) + V_0$ and $Rad(V) \cap V_0 = \{0\}$. The corresponding decomposition of $V$ is called a Levi decomposition. We remark that a Levi subalgebra is semisimple if it exists.

<u>Theorem 4.1</u>: *(Levi decomposition)*
*Let $V$ be an $n$-Lie algebra over $K$ and $n \geq 3$. Then $V$ admits a Levi subalgebra.*

Let us describe the idea of the proof of Theorem 4.1. By induction on the dimension of the radical of $V$ we will reduce the proof to the case that $Rad(V)$ is a minimal ideal of $V$. Then we consider the Lie algebra $Der(V)$. First, we make the Levi decomposition: $Der(V) = Rad(Der(V)) + L_0$, where $L_0$ is a Levi subalgebra of $Der(V)$. With the help of the Lie algebra homomorphism $\gamma$ defined in Theorem 2.9 we decompose $L_0$ into a sum of two ideals $L_1$ and $L_2$: $L_0 = L_1 + L_2$, where $L_1$ is a Levi subalgebra of $Ker\gamma$. Then we decompose $Rad(Der(V))$ into the sum of $N$ and $A$ where $N$ is the ideal of $Der(V)$ consisting of all nilpotent endomorphisms in $Rad(Der(V))$ and $A$ is an abelian subalgebra consisting of semisimple endomorphisms of $V$ and $[A, L_0] = \{0\}$. Since $V$ is a completely reducible $(A + L_0)$-module and $I$ is a submodule of $V$, there exists a submodule $V_0$ such that $V = I + V_0$ (vector space direct sum). In case $A + L_1 \neq \{0\}$, we show that $V_0$ is the subspace of elements in $V$ killed by all elements in $A + L_1$ which shows that $V_0$ a Levi subalgebra. In case $A + L_1 = \{0\}$, we discuss the possibilities of the appearence of $I$ in the isotypic components of $[V_0, \cdots, V_0]$. By using the decomposition of the $n$-fold wedge product of the natural $so(n, K)$-module $K^{n+1}$, we can show that either $V_0$ is a Levi subalgebra or there exists a nonzero $n$-Lie algebra homomorphism from $V/Rad(V)$ to $V$ whose image gives a Levi subalgebra.