Remark 1.2.1: If K is algebraically closed, then we mean by the vector product on K^{n+1} $(n > 2)$ the n-Lie algebra (K^{n+1}, b, f), where the determinant form f is normalized by $f(e_1, \cdots, e_n) = 1$ ($\{e_1, \cdots, e_{n+1}\}$ is the canonical base of K^{n+1}) and b is the bilinear form whose associated matrix relative to $\{e_1, \cdots, e_{n+1}\}$ of K^{n+1} is given by

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \text{or} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix},$$

according as n is odd or even, for some appropriate unit matrix I.

In the following we want to determine the group $Aut(V)$ of automorphisms of the vector product $V = (K^{n+1}, b, f)$. By Proposition 1.2.3 an automorphism τ of the vector space K^{n+1} is an automorphism of the vector product if and only if

$$b(\tau v, \tau w) = \det\tau \cdot b(v, w), \tag{1.13}$$

As a result of this we have that if τ is an automorphism of V, then $(\det \tau)^{n-1} = 1$ by choosing a base of V. Let $\Omega := \{\alpha \in K \mid \alpha^{n-1} = 1\}$.

Let $\tau \in Aut(V)$ and set $\sigma = \tau/\sqrt{\det \tau}$. Clearly $b(\sigma v, \sigma w) = b(v, w)$, that is, $\sigma \in O(K^{n+1}, b)$, while $\det\sigma = \det\tau/(\sqrt{\det\tau})^{n+1} = \pm 1$, according as $\det\sigma$ is a square in Ω or not. Therefore

$$\begin{aligned} Aut(V) \;\subseteq\; & \{\alpha\,\sigma \mid \alpha \in \Omega,\ \sigma \in SO(K^{n+1}, b)\} \cup \\ & \{\pm\sqrt{\beta}\,\sigma \mid \beta \in \Omega \setminus \Omega^2,\ \sigma \in O(K^{n+1}, b),\ \det\tau = -1\}. \end{aligned} \tag{1.14}$$

Since the multiplication by an element in Ω belongs to $Aut(V)$ and $SO(K^{n+1}, b) \subseteq Aut(V)$ by (1.13), the first set on the right side of (1.14) is included in $Aut(V)$. If n is even, every element in Ω is a square, then

$$Aut(V) = \{\alpha\,\sigma \mid \alpha \in \Omega,\ \sigma \in SO(K^{n+1}, b)\}. \tag{1.15}$$

Now let n be odd. For any non-square $\beta \in \Omega$ and any $\sigma \in O(K^{n+1}, b)$ with $\det\sigma = -1$, the endomorphism $\sqrt{\beta}\sigma$ is an automorphism of V. Indeed, because of $\det(\sqrt{\beta}\sigma) = \sqrt{\beta}^{n+1}\det\sigma = (-\beta)(-1) = \beta$ we have

$$b(\sqrt{\beta}\sigma u, \sqrt{\beta}\sigma v) = \beta\, b(\sigma u, \sigma v) = \beta\, b(u, v) = \det(\sqrt{\beta}\sigma)\, b(u, v).$$

According to (1.13), $\sqrt{\beta}\sigma$ is an element in $Aut(V)$. Therefore we have proved

$$\begin{aligned} Aut(V) \;=\; & \{\alpha\,\sigma \mid \alpha \in \Omega,\ \sigma \in SO(K^{n+1}, b)\} \cup \\ & \{\pm\sqrt{\beta}\,\sigma \mid \beta \notin \Omega^2,\ \sigma \in O(K^{n+1}, b),\ \det\tau = -1\}. \end{aligned} \tag{1.16}$$