Paramodular forms on $GSp_2(\mathbb{A})$

Dedicated to Professor Hiroyuki Yoshida on his 60th birthday

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By the theta lift from $GO(2,2)$, we give a construction of generic automorphic forms on $GSp_2(\mathbb{A})$ which are fixed by paramodular groups. We can say our construction is canonical, by the reason why they are not vanishing and why they have acceptable local L-factors and ε-factors at bad primes.

Introduction.

For a given $GSp_2(\overline{\mathbb{Q}_l})$-valued motivic Galois representation ρ of $\mathfrak{G}_{\mathbb{Q}} = Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, an automorphic representation π_ρ of $GSp_2(\mathbb{A}_{\mathbb{Q}})$ is expected to correspond. Yoshida [?] conjectured that for every abelian surface A over $\mathbb{Q}$, there exists a holomorphic Siegel modular form F_A of weight 2 of a suitable level whose spinor L-function $L^{spin}(s, F_A)$ equals the Hasse-Weil zeta function of A. This is one of such type of conjecutre, and the evidences are given by the Yoshida lift, a theta lift from $GO(4)$ to $GSp(2)$. However the thoery of holomorphic newform was not established in the $GSp(2)$-case. So, it is hard to describe the level of the conjectural form F_A. Recently, B. Roberts and R. Schmidt [?] have established the local generic newform theory in the trivial central character case as follows. Let $k_{\mathfrak{p}}$ be a nonarchimedean local field and $\mathfrak{p}$ be the prime ideal of the ring of integer $\mathfrak{o}$ of $k_{\mathfrak{p}}$. Let $\pi = W(\pi, \psi_{1,\ 1})$ be a local generic representaion., i.e., π has a Whitakker model. Suppose that the central character η_π is trivial. Then there exists the unique (up to scalar) newvector $W^{new} \in W(\pi, \psi_{1,\ 1})$ fixed by the local paramodular group

$$K_{\mathfrak{p}}(N_\pi) = \{u \in \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{p}^{-N_\pi} & \mathfrak{o} \\ \mathfrak{p}^{N_\pi} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^{N_\pi} & \mathfrak{p}^{N_\pi} & \mathfrak{o} & \mathfrak{p}^{N_\pi} \\ \mathfrak{p}^{N_\pi} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \end{bmatrix} \mid \nu(u) \in \mathfrak{o}^\times\} \tag{0.1}$$

of conductor $N_\pi \in \mathbb{Z}_{\geq 0}$, where $\nu(u)$ is the similitude norm of $u \in GSp_2(k_{\mathfrak{p}})$. The non-negative integer N_π is intrerpreted as the conductor of π, and characterized by the local functional equation $L(1-s, \tilde{\pi}) = \pm N_\pi^{s-\frac{1}{2}} L(s, \pi)$ with the contragredient $\tilde{\pi}$ of π. So, by this beautiful newform theory, it is natural to provide the following parrarel conjecture.

CONJECTURE 1 *For every abelian surface A over $\mathbb{Q}$, there exists a generic Siegel modular form F_A of weight $(2,0)$ fixed by $\prod_p K_p(c(\rho_{A,p}))$ with trivial central character such that $L^{spin}(s - \frac{1}{2}, F_A)_v = L(s, H^1_{et}(A, \mathbb{Q}_l))_v$ at every place v. Here $c(\rho_{A,p})$ is the conductor of the local Galois representation $\rho_{A,p}$ associated to A.*

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REMARK 0.1 *We don't add the condition of the functional sign + as in Yoshida's original conjecture.*

Of course, if this conjecture is true, then Hasse's conjecrture on the functional equation is true in precise form given by Serre. In this paper, we will give the evidences for this conjecture by the theta lift from $GO(2,2)$ in the following cases (c.f. Theorem **??**

(A) Let L be a real quadratic field with the discriminant d_L, and f be a Hilbert newform of multiple weight $(2,2)$ of level $\mathfrak{n}$. Then there exists $F \in M_{2,0}(\prod_p K_p(ord_p(N_{L/\mathbb{Q}}(\mathfrak{n})d_L)))$ with $L^{spin}(s,F)_v = L(s,f)_v$, at every place v of Q.

(B) Let f_1, f_2 be elliptic newforms of weight 2 with level N_1, N_2. Then there exists $F \in M_{2,0}(\prod_p K_p(ord_p(N_1N_2))$ with $L^{spin}(s,F)_v = L(s,f_1)_v L(s,f_2)_v$ at every place v of Q.

(B) means that all jacobian varieties of elliptic modular curves of genus 2 are Siegel modular in the generic sence. (A) means all motives of Hilbert modular forms over a real quadratic field of weight $(2,2)$ are also Siegel modular, e.g, jacobian of Shimura curves obtained by indefinite quaternion algebras, and abelian surface with complex multiplication of quartic CM-fields.

In this paper, more generally than the cases described as above, we will give the construction of Hilbert-Siegel modular forms fixed by paramodular groups in the trivial central character case. Further in the nontrivial central character case also, we give good modular forms semi-stable on *'semi-paramodular groups* (c.f. Definition **??**, and provide a conjecture (p.p) for the case. We consider our construction is canonical, from the reason why they are not vanishing and why they have acceptable local L-factors and ε-factors at bad primes. Our construction will be applied for explicit research in a nearly future.

On the other hand, we remark that holomorphic discrete series representation of $GSp(2)$ cannot cover all the L-functions of abelian surfaces. Indeed, suppose that one of f_1, f_2 in (B) above doesn't correspond to automorphic representations of definite quaternion algebras in the sence of Jacquet-Langlands [?] (e.g., one of elliptic curves E_1, E_2 corresponds to unramified or ramified principal series representation of $GL_2(\mathbb{Q}_p)$ at every p). In this case, we cannot use the Yoshida lift to construct a holomorphic modular form F with $L_S^{spin}(s,F) = L_S(s,f_1)L_S(s,f_2)$ (S is a set of finitely many primes and L_S means the exception of Euler factors of S). Further, suppose that there exists such a holomorphic F. Then, the standard L-function

$$L_S^{st}(s,F) = \zeta_S(s)^{-1} L_S(s+1, \rho_{f_1} \otimes \rho_{f_2}, \wedge^2) = \zeta_S(s) L_S(s, f_1 \times f_2) \tag{0.2}$$

has, at least, a simple pole at $s = 1$ where ρ_{f_i} is the Galois representation of $\mathfrak{G}_{\mathbb{Q}}$ associated to f_i. According to Kudla-Rallis-Soudry [?], F is liftable to an automorpchic form of a compact orthogonal group of rank 4: i.e., $Sp(2) \to O(4)$ which is the converse direction of the Yoshida lift. By this reason, we cannot hope holomorphic Siegel modular forms for such abelian surfaces. (Contrary to this, the theta lift from $O(2,2)$ can provide the desired generic Siegel modular forms, which is not holomorphic.) Following (**??**), we put

$$Z_S(s, \rho_A) := \zeta_S(s)^{-1} L_S(s+1, \rho_A, \wedge^2).$$

If $Z_S(1,\rho_A) \in \mathbb{C}^\times$, then the modularity of A also comes down from (conjectual) generic cuspidal representation of $GL_4(\mathbb{A}_\mathbb{Q})$ by the transfer lift, and we hope this value is interpreted as a size of some Selmer group (and is not zero) like in the elliptic modular case. We remark that it is possible to give the correponding form F_A by theta lifts from orthogonal groups of rank 4, if and only if $-ord_{s=1}Z_S(s,\rho_A,\chi) \geq 1$ for some quadratic charqacter χ. Indeed,

(A) If A is isogeneous to a Hilbert motief, i.e., $L(s,H^1_{et}(A,\mathbb{Q}_l)) = L(s,f)$ for a Hilbert cuspform f over L, then $-ord_{s=1}Z_S(s,\rho_A,\chi_{L/\mathbb{Q}}) \geq 1$. Furhter, if and only if $-ord_{s=1}Z_S(s,\rho_A,\chi_{L/\mathbb{Q}}) \geq 2$, A is isogeneous to the Weil restriction $\mathrm{Res}_{L/\mathbb{Q}}(E)$ of the definition of the field, for a Q-curve E defined over L, and F_A is a noncuspform.

(B) If A is isogeneous to a product of elliptic curves $E_1 \times E_2$, then $-ord_{s=1}Z_S(s,\rho_A) \geq 1$. Further, if and only if $-ord_{s=1}Z_S(s,\rho_A) \geq 2$, E_1 is isogeneous to E_2, and F_A is a noncuspform.

Since the Yoshida conjecture is (and conjecture 1 is also, perhaps) still a difficult problem, to show the inverse directions of above cases may be a reasonble problem. Related to the case (A), we have the following topic, which is the reason why we attend the condition $\eta_{F_A} = 1$ in the conjecture 1. There exists a pair of cuspform F and a non-cuspform E such as $L^{spin}(s,F) = L^{spin}(s,E) = L(s,H^1_{et}(Jac(C),\mathbb{Q}_l))$ for the hyper elliptic curve $C: y^2 = x^5 - x$ (c.f. [?]) although F doesn't belongs to CAP representations. We will discuss this accidental case and give an answer Theorem **??** for the question when such an accident occurs, and give a characterization of non CAP cuspidal representation by standard L-function.

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Notation: For an abel group H, $\mathfrak{X}(H)$ denotes the group of quasi-characters of H and $\mathfrak{X}^1(H)$ does that of unitary characters. For an adelized group $G_\mathbb{A}$, we denote by $\mathcal{A}(G_\mathbb{A})$ the set of automorphic forms on $G_\mathbb{A}$, and by $\Pi(G_\mathbb{A})$ that of irreducible automorphic representations. If a function f on a group G satisfies $f(gu) = \chi(u)f(g)$, $g \in G, u \in \mathbb{K}$ for a fixed character χ on a certain subgroup $\mathbb{K}$ of G, we say f is '*χ-semi stable on* $\mathbb{K}$' or '*semi stable on* $\mathbb{K}$'. If χ is trivial, we say *stable on* $\mathbb{K}$, simply. If $\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GSp_m(A) \subset GL_{2m}(A)$ with a,b,c,d of size m, we shall write $a = a_\alpha, b = b_\alpha, c = c_\alpha$, and $d = d_\alpha$. For a ring A and an element $x \in A$, we denote the following elements in $M_2(A)$ in this way:

$$n_x = n(x) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, \alpha_x = \alpha(x) = \begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix}, w_x = w(x) = \begin{bmatrix} 0 & 1 \\ x & 0 \end{bmatrix}, v_x = v(x) = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}.$$

For a symmetric matrix $Q \in M_n(A)$, and $X \in M_{n,m}(A)$, we write $Q[X] = {}^tXQX \in M_m(A)$. By u_θ with $\theta \in \mathbb{R}$, we mean the element $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \in SO_2(\mathbb{R})$.

1 theta lift from $\mathcal{A}(GL_2(\mathbb{A}))^2$ to $\mathcal{A}(GSp_2(\mathbb{A}))$

Let k be a totally real field of degree d with ring $\mathfrak{o}$ of integers. Fix $\psi = \psi_k \in \mathfrak{X}(k\backslash\mathbb{A})$ by

$$\begin{cases} \psi_\infty(z) = \prod_{j=1}^d \exp(\pi i z_j) & \text{for } z = (z_1,\ldots,z_d) \in k_\infty. \\ \psi_\mathfrak{p}(z) = \exp(-\pi i \cdot Tr_{k_\mathfrak{p}/\mathbb{Q}_p}(\text{the fractional part of } z)) & \text{for } z \in k_\mathfrak{p}. \end{cases}$$

We require the following conditions for unitary $\sigma_1, \sigma_2 \in \Pi(GL_2(\mathbb{A}))$:

1) At least, one of σ_1 or σ_2 is cuspidal.

2) σ_1 and σ_2 have an idetical central character η.
3) At each archimedean place ∞_j of k, both of $\sigma_{1\infty_j}$ and $\sigma_{2\infty_j}$ are discrete series representation with an identical lowest weight $\kappa_j(\geq 1)$.

Automorphic forms of $GL_2(\mathbb{A})$: (??條件 For an fractional ideal $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{o}$, we define the congruence subgroups of $GL_2(\mathfrak{o})$ by

$$\Gamma_0(\mathfrak{a}) = \{\gamma \in GL_2(\mathfrak{o}) \mid c_\gamma \in \mathfrak{a}\}; \ \Gamma_{00}(\mathfrak{a}, \mathfrak{b}) = \{\gamma \in \Gamma_0(\mathfrak{a}) \mid b_\gamma \in \mathfrak{b}\}.$$

At each finite place $\mathfrak{p}$ of k, we define the local congruence subgroups $\Gamma_0(\mathfrak{p}^m), \Gamma_{00}(\mathfrak{p}^n, \mathfrak{p}^m) \subset GL_2(\mathfrak{o}_\mathfrak{p})$, similarly. When it is clear that we do a local discussion, we write them as

$$\Gamma_0(m) = \{\gamma \in GL_2(\mathfrak{o}_\mathfrak{p}) \mid c_\gamma \in \mathfrak{p}^m\}; \ \Gamma_{00}(n, m) = \{\gamma \in \Gamma_0(m) \mid b_\gamma \in \mathfrak{p}^n\}.$$

Let $W(\psi_v, \sigma_v)$ be the space of Whittaker functions belonging to σ_v with respect to ψ_v. According to p.71 of Weil [?], we can choose the following Whitakker function belonging to $W(\sigma_{i\infty_j}, \psi_{\infty_j}) = W(\tilde{\sigma}_{i\infty_j}, \psi_{\infty_j})$,

$$W_{i\infty_j}(zn_x\alpha_y u_\theta) = (\frac{z}{|z|})^r \psi_{\infty_j}(x) Wh_{(\frac{\kappa_j}{2}, \frac{\kappa_j-1}{2})}(|y|)\exp(i\kappa_j\theta)$$

for $x \in \mathbb{R}, y, z \in \mathbb{R}^\times, u_\theta \in SO_2(\mathbb{R})$ and $r = 1$ or 0. Here Wh denotes Whittaker's function in the original sense associated to $(\frac{\kappa_j}{2}, \frac{\kappa_j - 1}{2})$. At finite place $v = \mathfrak{p}$, $W(\sigma_\mathfrak{p}, \psi_\mathfrak{p})$ has a newvector which is '*semi-stable on* $\Gamma_0(\mathfrak{e})$' for the conductor $\mathfrak{e}$ of $\sigma_\mathfrak{p}$. Let $\mathfrak{p}^\delta = \delta_{k_\mathfrak{p}}$ be the local different of $k_\mathfrak{p}/\mathbb{Q}_p$ (then, $\psi_\mathfrak{p}(\mathfrak{p}^{-\delta}) = 1, \psi_\mathfrak{p}(\mathfrak{p}^{-\delta-1}) \neq 1$). Here we state a lemma:

LEMMA 1.1 *If* $W \in W(\psi_\mathfrak{p}, \sigma_\mathfrak{p})$ *is stable on* $N(-\delta) := \{n_x \mid x \in \mathfrak{p}^{-\delta}\}$, *then* $W(\alpha(\varpi^m)) = 0$ *for* $m \in \mathbb{Z}_{<0}$.

This lemma is obtained easily by

$$W(\alpha(\varpi^m)n(x)) = W(n(\varpi^m x)\alpha(\varpi^m)) = \psi_\mathfrak{p}(\varpi^m x)W(\alpha(\varpi^m)). \tag{1.3}$$

We take $W_{1\mathfrak{p}} \in W(\tilde{\sigma}_{1\mathfrak{p}}, \psi_\mathfrak{p})$ and $W_{2\mathfrak{p}} \in W(\sigma_{2\mathfrak{p}}, \psi_\mathfrak{p})$ so that

$$W_{1\mathfrak{p}}(hu_1) = \eta_\mathfrak{p}^{-1}(d_{u_1})W_{1\mathfrak{p}}(h), \ W_{2\mathfrak{p}}(hu_2) = \eta_\mathfrak{p}(a_{u_2})W_{2\mathfrak{p}}(h), \tag{1.4}$$

for $u_i \in \Gamma_{00}(\mathfrak{e}_i + \delta, -\delta)$. We set

$$f_1(h) = \sum_{t \in k^\times} \prod_v W_{1v}(\alpha_t h); \ f_2(h) = \sum_{t \in k^\times} \prod_v W_{2v}(\alpha_t h), \ \ h \in GL_2(\mathbb{A}), \tag{1.5}$$

both of which are automorphic forms on $GL_2(\mathbb{A})$ by Jacquet, Langlands [?].

Schwartz functions: We set the quadratic form in $M_2(k)$ by $Q[x] = 2\det(x)$, which induces the bilinear form $\langle x, y\rangle = tr(xy^\iota)$ where ι is the main involution of $M_2(k)$. We denote by also $Q \in M_4(\mathbb{Q})$ the symmetric (integral) matrix associated to the quadratic form. Put $H(k) = GL_2(k)^2$ and $H^1(k) = \{h = (h_1, h_2) \in H(k) \mid \det h_1 = \det h_2\}$. Define the right action ρ of $H(k)$(resp. $H(\mathbb{A})$) on $M_2(k)$ (resp. $M_2(\mathbb{A})$) by

$$\rho((h_1, h_2))x = h_1^{-1}xh_2, \ \ x \in M_2(k).$$

For the theta lift, we prepare Schwartz function $\varphi_v \in \mathcal{S}(M_2(k_v)^2)$ *'corresponding to'* the above pair W_{1v}, W_{2v} as follows.

(*At j-th archimedean place* ∞_j) We choose two polynomials of $M_2(\mathbb{R})$

$$P_1(x) = \langle x, \begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix} \rangle = i(d_x - a_x) - (b_x + c_x),\ P_2(x) = \langle x, \begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix} \rangle = i(a_x + d_x) + (c_x - d_x),$$

so that $P_1(u_{\theta_1}^{-1} x u_{\theta_2}) = e^{-i(\theta_1+\theta_2)} P_1(x)$, $P_2(u_{\theta_1}^{-1} x u_{\theta_2}) = e^{i(\theta_1-\theta_2)} P_2(x)$. Define $\varphi_{\infty_j} \in \mathcal{S}(M_2(\mathbb{R})^2, \mathbb{C}[s_1, s_2])$ which is homogeneous polynomial of s_1, s_2 by

$$\varphi_{\infty_j}(x_1, x_2) = \exp(-\pi\, \mathrm{tr}(R[x_1, x_2])) P_1(s_1 x_1 + s_2 x_2)^a \times \begin{cases} P_2(s_2 x_1 - s_1 x_2)^b & \text{if } \kappa_{1j} \le \kappa_{2j}, \\ \overline{P}_2(s_2 x_1 - s_1 x_2)^b & \text{otherwise,} \end{cases}$$

where $a = \frac{\kappa_{1j}+\kappa_{2j}}{2}$ and $b = \frac{\kappa_{1j}-\kappa_{2j}}{2}$. Here $R = {}^tR \in M_4(k)$ is chosen so that $R[x_i] = a_{x_i}^2 + b_{x_i}^2 + c_{x_i}^2 + d_{x_i}^2$ and x_i is regarded as a line vector (so $R[x_1, x_2] = {}^t(R[x_1, x_2]) \in M_2(\mathbb{R})$). R is an Hermite's minimal majorant of Q, i.e., $RQ^{-1}R = Q$ (c.f. §1 of Oda [?]).

(*At finite place* $\mathfrak{p}$) We write $\mathfrak{o}$ for $\mathfrak{o}_\mathfrak{p}$ simply, and fix an uniformizer $\varpi = \varpi_\mathfrak{p} \in \mathfrak{p}$. Let $\mathfrak{c}_i$ be the conductor of $\sigma_{i\mathfrak{p}}$, and $\mathfrak{c} = \mathfrak{c}_1 + \mathfrak{c}_2$. Let $\mathfrak{f} = c(\eta_\mathfrak{p})$ be the conductor of the central character $\eta_\mathfrak{p}$ of $\sigma_{1\mathfrak{p}}, \sigma_{2\mathfrak{p}}$. In the case of $\mathfrak{f} = 0$, we define

$$\varphi_\mathfrak{p}(x_1, x_2) = \text{the characteristic function of } \begin{bmatrix} \mathfrak{p}^{\mathfrak{c}_2} & \mathfrak{p}^{-\delta} \\ \mathfrak{p}^{\mathfrak{c}+\delta} & \mathfrak{p}^{\mathfrak{c}_1} \end{bmatrix} \oplus \begin{bmatrix} \mathfrak{o} & \mathfrak{p}^{-\delta} \\ \mathfrak{p}^{\delta} & \mathfrak{o} \end{bmatrix}.$$

In the case of $\mathfrak{f} > 0$, we define

$$\varphi_\mathfrak{p}(x_1, x_2) = \begin{cases} \eta_\mathfrak{p}(\varpi^\delta b_{x_1}), & \text{if } (x_1, x_2) \in \begin{bmatrix} \mathfrak{p}^{\mathfrak{c}_2} & \mathfrak{p}^{-\delta}\mathfrak{o}^\times \\ \mathfrak{p}^{\mathfrak{c}+\delta} & \mathfrak{p}^{\mathfrak{c}_1} \end{bmatrix} \oplus M_2(\mathfrak{o}), \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathfrak{p}^\delta$ is the different of $k_\mathfrak{p}$ over $\mathbb{Q}_p$. Of course, $\varphi_\mathfrak{p}$ is the characteristic function of $M_2(\mathfrak{o}) \oplus M_2(\mathfrak{o})$ at almost all $\mathfrak{p}$. We define a congruence subgroup $\mathbb{U}_{m,n}$ of $H^1(k_\mathfrak{p})$ for $m, n \in \mathbb{Z}_{\ge 0}$ by

$$\mathbb{U}_{m,n} = \{(h_1, h_2) \in H^1(k_\mathfrak{p}) \mid h_1 \in \Gamma_{00}(m+\delta, -\delta), h_2 \in \Gamma_{00}(n+\delta, -\delta)\}. \tag{1.6}$$

For $(h_1, h_2) \in H^1(k_\mathfrak{p})$ and $(u_1, u_2) \in \mathbb{U}_{\mathfrak{c}_1, \mathfrak{c}_2}$, it holds

$$\rho(h_1 u_1, h_2 u_2)\varphi_v(x_1, x_2) f_1(h_1 u_1) f_2(h_2 u_2) = \rho(h_1, h_2)\varphi_v(x_1, x_2) f_1(h_1) f_2(h_2). \tag{1.7}$$

But this equality does not hold for (u_1, u_2) belonging to $\mathbb{U}_{\mathfrak{c}_1-1, \mathfrak{c}_2}, \mathbb{U}_{\mathfrak{c}_1, \mathfrak{c}_2-1}, \Gamma_{00}(\mathfrak{c}_1+\delta, -\delta-1) \times \Gamma_{00}(\mathfrak{c}_2+\delta, -\delta)$, nor $\Gamma_{00}(\mathfrak{c}_1+\delta, -\delta) \times \Gamma_{00}(\mathfrak{c}_2+\delta, -\delta-1)$. By this condition, $\varphi_\mathfrak{p}$ is the key hole for the non-vanishing of the theta lift, as seen later. By this reason, we said $\varphi_\mathfrak{p}$ is *'corresponding to'* the pair $W_{1\mathfrak{p}}, W_{2\mathfrak{p}}$.

Theta lift: As in section 5.1 of Harris, Kudla [?], we extend Weil representation $r = \otimes_v r_v$ of $GSp_2 \times GO_4$ associated to the additive character $\psi = \psi_k$, and define the theta lift by

$$\theta(g; f_1 \boxtimes f_2) = \int_{\mathbb{A}^\times H^1(k) \backslash H^1(\mathbb{A})} \sum_{x_i \in M_2(k)} (r(g, h)\varphi)(x_1, x_2) f_1 \boxtimes f_2(hh') dh. \tag{1.8}$$

Here, associated to each element $g \in GSp_2(\mathbb{A})$, we choose $h' = (h_1', h_2') \in GL_2(\mathbb{A})^2$ so that $\nu(g) = \det(h_1')\det(h_2')^{-1}$. We embed $\mathbb{A}^\times \ni a \longmapsto (a, a) \in H^1(\mathbb{A})$.

PROPOSITION 1.2 *At j-th archimedean place ∞_j of k, the highest weight of the representation of $U_2(\mathbb{C}) \subset Sp_2(\mathbb{R})$ associated to $\theta(g; f_1 \boxtimes f_2)$ is $(\frac{\kappa_{1j}+\kappa_{2j}}{2}, -|\frac{\kappa_{1j}-\kappa_{2j}}{2}|)$.*

PROOF. We treat only the case of $\kappa_{1j} \leq \kappa_{2j}$, since the other case is done in the symmetric way. For the simplicity, we write $\psi(z) = \exp(\pi i z)$, and $P_1(x;s) = P_1(s_1x_1 + s_2x_2), P_2(x;s) = P_2(s_2x_1 - s_1x_2)$ with $x = (x_1, x_2), s = {}^t(s_1, s_2)$. Let $\varphi_{a,b}(x;s) = \varphi_{\infty_j}(x)$. Let r' denote the restriction of the Weil representaion r_{∞_j} to $Sp_2(k_{\infty_j}) \simeq Sp_2(\mathbb{R})$. It is sufficient to show that, for every $u \in U_2(\mathbb{C})$,

$$r'(u)\varphi_{a,b}(x;s) = \det(u)^{-b}\varphi_{a,b}(x;us). \tag{1.9}$$

Remark that $\varphi_{a,b}(x;us) = \varphi_{a,b}(x;s)\mathrm{sym}_{a+b}(u)$, since they are homogeneous polynoimals of s of degree $a+b$. We can decompose $u = \begin{bmatrix} A & B \\ -B & A \end{bmatrix} = \begin{bmatrix} {}^tB^{-1} & A \\ 0 & B \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & B^{-1}A \\ 0 & 1 \end{bmatrix}$, if $\det(B) \neq 0$. If (**??**) is shown for all such u, then (**??**) holds for evrey $u \in U_2(\mathbb{C})$ by continuity of r' on the L^2-norm. In the case of $a = b = 0$, by the above decomposition of u and the well known formula

$$\int_{\mathbb{R}} \exp(-\pi x^2 - 2\pi\sqrt{-1}\lambda x)dx = \exp(-\pi\lambda^2), \ \ \lambda \in \mathbb{C},$$

we can derive from $r'(u)\varphi_0$ the trivial character on

$$\left\{ \begin{bmatrix} u_1 & & v_1 & \\ & u_2 & & v_2 \\ -v_1 & & u_1 & \\ & -v_2 & & u_2 \end{bmatrix} \;\middle|\; |u_1|^2 + |v_1|^2 = |u_2|^2 + |v_2|^2 = 1 \right\} \simeq U_1(\mathbb{C})^2$$

which is a maximal torus of $U_2(\mathbb{C})$, and derive the trivial representation of $A + 0i \in O_2(\mathbb{R}) \subset U_2(\mathbb{C})$. This implies $r'(u)$ is trivial. For general a, b, we shall show

$$\Psi^*_{a,b}(x;s) = \det(u)^{-b} r'(\begin{bmatrix} {}^tB & {}^tBAB^{-1} \\ 0 & B^{-1} \end{bmatrix})\varphi_{a,b}(x;us) \tag{1.10}$$

which is equivalent to (**??**). Here $\Psi^*_{a,b}(x;s)$ is the Fourier transformation of $\Psi_{a,b}(x;s) = r'(\begin{bmatrix} 1 & -B^{-1}A \\ & 1 \end{bmatrix})\varphi_\kappa(x;s)$ with respect to ψ. Define the differential operator Δ_1, Δ_2 acting on $\mathcal{S}(M_2(\mathbb{R})^2; \mathbb{C}[s_1, s_2])$ by

$$\Delta_1 := i(2\pi)^{-1}P_1(s_1\mathfrak{D}_1 + s_2\mathfrak{D}_2), \ \Delta_2 := (2\pi i)^{-1}P_2(s_2\mathfrak{D}_1 - s_1\mathfrak{D}_2), \ \ \mathfrak{D}_i = \begin{bmatrix} \partial/\partial a_{x_i} & \partial/\partial b_{x_i} \\ \partial/\partial c_{x_i} & \partial/\partial d_{x_i} \end{bmatrix}.$$

By acting $\Delta_1^a\Delta_2^b$ on $\Psi^*_{0,0}(x;s) = r'(\begin{bmatrix} {}^tB & {}^tBAB^{-1} \\ 0 & B^{-1} \end{bmatrix})\varphi_{0,0}(x;us)$, we are going to obtain (**??**) for general a, b. Using $\Delta_iP_j(x,s) = 0$ for $i, j = 1, 2$, we can see easily

$$\Delta_1^a\Delta_2^b\psi(2\langle x_1, y_1\rangle + 2\langle x_2, y_2\rangle) = P_1(y;s)^aP_2(y;s)^b\psi(2\langle x_1, y_1\rangle + 2\langle x_2, y_2\rangle), \ \ y = (y_1, y_2).$$

So, we obtain the left hand side of (**??**)

$$\Delta_1^a\Delta_2^b\Psi_0^*(x) = \int_{M_2(\mathbb{R})^2} P_1(y;s)^aP_2(y;s)^b\Psi_0(y)\psi(2\langle x_1, y_1\rangle + 2\langle x_2, y_2\rangle)dy_1dy_2$$

by exchanging Δ_j and the integrand. On the other hand, the right hand side of (**??**) is obtained by the induction on a, b. Assume (**??**) holds for a, b. Then, using $\Delta_1P_j(x;gs) = 0, j = 1, 2$ for any $g \in$

$GL_2(\mathbb{R})$ and $\Delta_1\psi(tr(\sigma\cdot Q[x])=\psi(tr(\sigma\cdot Q[x])P_1(x;\sigma s)$; $\Delta_1\varphi_0(xh)=P_1(x;ih^ths)\varphi_0(xh)$ for any ${}^t\sigma=\sigma\in M_2(\mathbb{R}), h\in GL_2(\mathbb{R})$, the right hand side of (??) for $a+1,b$ is obtained in the following way.

$$\begin{aligned}&\det(B)^2(\Delta_1\psi(tr({}^tBA\cdot Q[x]))\varphi_{a,b}(x^tB;us)+\det(B)^2\psi(tr({}^tBA\cdot Q[x])(\Delta_1\varphi_{a,b}(x^tB;us))\\ =&\det(B)^2\psi(tr({}^tBA\cdot Q[x]))P_1(x;{}^tBAs)\varphi_{a,b}(x^tB;us)+\det(B)^2\psi(tr({}^tBA\cdot Q[x]))P_1(x;i^tBBs)\varphi_{a,b}(x^tB;us)\\ =&\det(B)^2\psi(tr({}^tBA\cdot Q[x]))P_1(x^tB;(A+iB)s)\varphi_{a,b}(x^tB;us)=\det(B)^2\psi(tr({}^tBA\cdot Q[x]))\varphi_{a-1,b}(x^tB;us).\end{aligned}$$

Here we wtite $x^tB=(a_Bx_1+c_Bx_2,b_Bx_1+d_Bx_2)$. The right hand side of (??) for $a,b+1$ is also obtained in the following way. Let $\rho(x):={}^tu'$. Using $\Delta_2P_j(x;gs)=0, j=1,2$ for any $g\in GL_2(\mathbb{R})$ and $\Delta_2\varphi_0(x\sigma)=P_2(x;i\rho(\sigma)s)\varphi_0(x)$ for $\sigma={}^t\sigma\in GL_2(\mathbb{R})$,

$$\begin{aligned}&\det(u^{-b}B^2)\Big((\Delta_2\psi(tr({}^tBA\cdot Q[x]))\varphi_{a,b}(x^tB;us)+\psi(tr({}^tBA\cdot Q[x])(\Delta_2\varphi_{a,b}(x^tB;us))\Big)\\ =&\det(u^{-b}B^2)(\psi(tr({}^tBA\cdot Q[x]))P_2(x;\rho({}^tBA)s)\varphi_{a,b}(x^tB;us)+\psi(tr({}^tBA\cdot Q[x]))P_2(x;-i\rho({}^tBB)s)\varphi_{a,b}(x^tB;us))\\ =&\det(u^{-b}B^2)\psi(tr({}^tBA\cdot Q[x]))P_2(x^tB;\rho(A-iB)s)\varphi_{a,b}(x^tB;us)=\det(u^{-b-1}B^2)\psi(tr({}^tBA\cdot Q[x]))\varphi_{a,b-1}(x^tB;us).\end{aligned}$$

Here remark that $\rho(\overline{u})=(u')^{-1}=\det(u)^{-1}u$ at the last line. This completes the proof. $\square$

REMARK 1.3 *According to Przebinda [?], the irreducible representation π_{∞_j} associtated to F is a P_1-principal series reperesentation in the sence of p.904 of Moriyama [?] if $\kappa_{1j}=\kappa_{2j}$, and a discrete series representation otherwise. Hecne, by [?], it holds*

$$L^{spin}(s,\pi_{\infty_j})=4(2\pi)^{-2s}\Gamma(s+\frac{\kappa_{1j}-1}{2})\Gamma(s+\frac{\kappa_{2j}-1}{2})$$

which is equal to $L(s,\widetilde{\sigma}_{1\infty_j})L(s,\widetilde{\sigma}_{2\infty_j})$, up to constant.

In order to describe an local property of $\theta(g,f_1\boxtimes f_2)$ at finite place $\mathfrak{p}$, we introduce the following concept:

DEFINITION 1.1 ($\eta_\mathfrak{p}$-SEMISTABLE ON $K_\mathfrak{p}(\mathfrak{c},\mathfrak{f})$) *Define the semi-paramodular group $K_\mathfrak{p}(\mathfrak{c},\mathfrak{f})$ of level $(\mathfrak{c},\mathfrak{f})$ by*

$$K_\mathfrak{p}(\mathfrak{c},\mathfrak{f})=\begin{bmatrix}\mathfrak{o}&\mathfrak{o}&\mathfrak{p}^{-\mathfrak{c}}&\mathfrak{o}\\ \mathfrak{p}^{\mathfrak{c}}&\mathfrak{o}&\mathfrak{o}&\mathfrak{o}\\ \mathfrak{p}^{\mathfrak{c}+\mathfrak{f}}&\mathfrak{p}^{\mathfrak{c}}&\mathfrak{o}&\mathfrak{p}^{\mathfrak{c}}\\ \mathfrak{p}^{\mathfrak{c}}&\mathfrak{o}&\mathfrak{o}&\mathfrak{o}\end{bmatrix}\cap K_\mathfrak{p}(\mathfrak{c}). \tag{1.11}$$

For $\eta_\mathfrak{p}\in\widehat{k_\mathfrak{p}^\times}$, we say a function f on $GSp_2(k_\mathfrak{p})$ is $\eta_\mathfrak{p}$-semistable on $K_\mathfrak{p}(\mathfrak{c},\mathfrak{f})$, if f satisfies

$$f(zg)=\eta_\mathfrak{p}(z)f(g),\ \ f(gu)=\eta_\mathfrak{p}(d_1)f(g),\ \ g\in GSp_2(k_\mathfrak{p}),\ z\in k_\mathfrak{p}^\times,\ u=\begin{bmatrix}*&*&*&*\\ *&*&*&*\\ *&*&d_1&*\\ *&*&*&*\end{bmatrix}\in K_\mathfrak{p}(\mathfrak{c},\mathfrak{f}).$$

PROPOSITION 1.4 *At every finite $\mathfrak{p}$, $\theta(g,f_1\boxtimes f_2)$ is $\eta_\mathfrak{p}^{-1}$-semistable on $K_\mathfrak{p}(\mathfrak{c},\mathfrak{f})$.*

PROOF. Let $F=\theta(g;f_1\boxtimes f_2)$. In the case of $c(\eta_\mathfrak{p})=\mathfrak{f}=0$, the assertion is obtained by the same way as theorem 4.3 of [?]. In the case of $\mathfrak{f}>0$, it is easy to see that $F(g\cdot diag[z,y,z^{-1},y^{-1}])=\eta_\mathfrak{p}(z)F(g)$ for

$y, z \in \mathfrak{o}_\mathfrak{p}^\times$ from our setting of $\varphi_\mathfrak{p}$. So, it suffices to show that $F(zg) = F(g \cdot diag[1,1,z,z]) = \eta_\mathfrak{p}^{-1}(z)F(g)$. Put $h_z = (a_z, 1)$ with $z \in \mathfrak{o}_\mathfrak{p}^\times$. By the definition of the extended Weil representaion $r_\mathfrak{p}$, we have

$$\begin{aligned} F(g \cdot diag[1,1,z,z]) &= \int_{\mathbb{A}^\times H^1(k)\backslash H^1(\mathbb{A})} \sum_{x_i \in M_2(k)} (r(g, hh_z)\varphi)(x_1, x_2) f_1 \boxtimes f_2(hh_z)dh \\ &= \int_{\mathbb{A}^\times H^1(k)\backslash H^1(\mathbb{A})} \sum_{x_i \in M_2(k)} (r(g,h)\varphi)(x_1,x_2) f_1(h_1) \boxtimes f_2(h_2 a_z^{-1})dh \\ &= \int_{\mathbb{A}^\times H^1(k)\backslash H^1(\mathbb{A})} \sum_{x_i \in M_2(k)} (r(g,h)\varphi)(x_1,x_2)\eta_\mathfrak{p}^{-1}(z) f_1(h_1) \boxtimes f_2(h_2)dh. \end{aligned}$$

The last equality follows from (**??**). This completes the proof. □

Whittaker function of $\theta(g; f_1 \boxtimes f_2)$: For $c_1, c_2 \in k$, we define a character $\psi_{c_1,c_2}(u) = \psi(c_1 t + c_2 s) \in \mathfrak{X}(U(k)\backslash U(\mathbb{A}))$ of the maximal unipotent subgroup

$$U(\mathbb{A}) = \left\{ u = \begin{bmatrix} 1 & t & * & * \\ 0 & 1 & * & s \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -t & 1 \end{bmatrix} \in Sp_2(\mathbb{A}) \right\}$$

of $Sp_2(\mathbb{A})$. The global Whittaker function $W_F^{c_1,c_2}(g)$ of $F \in \mathcal{A}(GSp_2(\mathbb{A}))$ associated to ψ_{c_1,c_2} is defined by $\int_{U(k)\backslash U(\mathbb{A})} \overline{\psi_{c_1,c_2}(u)} F(ug)du$. We are going to calculate $W_F^{1,-1}(1)$. Write $f(h) = f_1(h_1) \boxtimes f_2(h_2)$ with $h = (h_1, h_2) \in H^1(\mathbb{A})$.

$$\begin{aligned} &\int_{U(k)\backslash U(\mathbb{A})} \overline{\psi_{1,-1}}(u) \Big(\int_{\mathbb{A}^\times H^1(k)\backslash H^1(\mathbb{A})} \sum_{(x_1,x_2) \in M_2(k)^2} r(u,h)\varphi(x_1,x_2) f(h)dh \Big) du \\ &= \int_{k\backslash\mathbb{A}} \overline{\psi}(t) \int_{N_P(k)\backslash N_P(\mathbb{A})} \int_{\mathbb{A}^\times H^1(k)\backslash H^1(\mathbb{A})} \sum_{(x_1,x_2)\in M_2(k)^2} \psi(\mathrm{tr}(B(u)(Q[x_1, tx_1 + x_2] - [\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}]))) \\ &\qquad \times r(1,h)\varphi(x_1, tx_1 + x_2) f(h)dh\, du, \end{aligned} \tag{1.12}$$

where we write $u = \begin{bmatrix} n(t) & \\ & {}^tn(-t) \end{bmatrix}\begin{bmatrix} 1 & B(u) \\ & 1 \end{bmatrix}$. Since f is rapidly decreasing, we can exchange the summation and integration at (**??**). All the pair $(x_1, x_2) \in M_2(k)^2$ which may contribute to $W_F^{1,-1}(1)$ should satisfy $\langle x_1, x_1 \rangle = \langle x_1, x_2 \rangle = 0$ and $\langle x_2, x_2 \rangle = -1$, that is, elements in $\rho(H(k))$-orbit of $(0, \alpha_{-1})$ or that of (v_{-1}, α_{-1}). However, the contribution of $\rho(H^1(k))$-orbit of $(0, \alpha_{-1})$ is found to be zero by replacing v_{-1} to 0 in (**??**) below. So, we will only see the contribution of the orbit of (v_{-1}, α_{-1}). Put

$$\begin{aligned} Z(k) &= \{h = (h_1,h_2) \in H^1(k) \mid \rho(h)v_{-1} = v_{-1}, \rho(h)\alpha_{-1} - \alpha_{-1} \in kv_{-1}\} = k^\times\{(n_x, n_y) \mid x, y \in k\}, \\ Z_0(k) &= \{h \in Z(k) \mid \rho(h)\alpha_{-1} = \alpha_{-1}\} = k^\times\{(n_x, n_{-x}) \mid x \in k\}. \end{aligned} \tag{1.13}$$

Then (**??**) is calculated as

$$\begin{aligned} &\int_{k\backslash\mathbb{A}} \overline{\psi}(t) \int_{Z_0(k)\backslash H^1(\mathbb{A})} r(\begin{bmatrix} n(t) & \\ & {}^tn(-t) \end{bmatrix}, h)\varphi(v_{-1}, \alpha_{-1}) f(h)dh\, dt \\ &= \int_{k\backslash\mathbb{A}} \overline{\psi}(t) \int_{Z_0(k)\backslash H^1(\mathbb{A})} r(1,h)\varphi(v_{-1}, tv_{-1} + \alpha_{-1}) f(h)dh\, dt. \end{aligned} \tag{1.14}$$

For every $t \in \mathbb{A}$, $s_t := (n_t, 1) = (1, n_t)$ uniquely as an element in $Z_0(\mathbb{A})\backslash Z(\mathbb{A})$ satisfies $\rho(s_t)\alpha_{-1} = tv_{-1} + \alpha_{-1}$. Noting

$$\overline{\psi}(t) r(1,h)\varphi(v_{-1}, tv_1 + \alpha_{-1}) f(h) = r(1, s_t h)\varphi(v_{-1}, \alpha_{-1})\overline{\psi}(t) f(h),$$

we can calculate (**??**) as

$$\int_{Z_0(k)\backslash H^1(\mathbb{A})}\int_{k\backslash\mathbb{A}} r(1,h)\varphi(v_{-1},\alpha_{-1})\overline{\psi}(t)f(s_th)dt\,dh - \int_{Z_0(\mathbb{A})\backslash H^1(\mathbb{A})} r(1,h)\varphi(v_{-1},\alpha_{-1})b_f(h)dh. \quad (1.15)$$

Here $b_f(h)$ is

$$\int_{Z_0(k)\backslash Z_0(\mathbb{A})}\left(\int_{k\backslash\mathbb{A}}\overline{\psi}(t)f(s_th'h)dt\right)dh' - \int_{k\backslash\mathbb{A}}\int_{k\backslash\mathbb{A}}\overline{\psi}(t+t')f_1(n(t)h_1)f_2(n(t')h_2)dtdt',$$

which is equal to the product $W_1(h_1)W_2(h_2)$ of Whittaker functions of f_1, f_2 associated to ψ. We are going to calculate each local factors of (**??**)

$$I_v = \int_{Z_0(k_v)\backslash H^1(k_v)} r_v(1,h)\varphi_v(v_{-1},\alpha_{-1})W_{1v}(h_1)W_{2v}(h_2)dh.$$

(The calculation of I_{∞_j}): Suppose $\kappa_{1j} = \kappa_{2j} = a$. Since $\rho(\alpha_y,\alpha_y n_x)(v_{-1},\alpha_{-1}) = \left(\begin{bmatrix} & y^{-1} \\ & \end{bmatrix}, \begin{bmatrix} 1 & x/y \\ & 1 \end{bmatrix}\right)$, the coefficient of s_1^a in $I_{\infty_j} \in \mathbb{C}[s_1,s_2]$ is, by substituting $s_1 = 1, s_2 = 0$, calculated as

$$\begin{aligned}
&2\int_{\mathbb{R}_{>0}} Wh_{-a/2,(a-1)/2}(y)^2\int_{\mathbb{R}}(-y)^{-a}\exp(-\pi(y^{-2}+2+(x/y)^2))\exp(2\pi i(x+iy))dxd^\times y \\
&= -2\exp(-2\pi)(-1)^a\int_{\mathbb{R}_{>0}}\exp(-\pi y^{-2})Wh_{-a/2,(a-1)/2}(y)^2\int_{\mathbb{R}}\exp(2\pi ix - \pi x^2y^{-2}))dxd^\times y \\
&- 2\exp(-2\pi)(-1)^{a+1}\int_{\mathbb{R}_{>0}} y\exp(-\pi(y^2-y^{-2}))Wh_{-a/2,(a-1)/2}(y)^2d^\times y.
\end{aligned}$$

This is not zero since $Wh_{-a/2,(a-1)/2}(y)^2 > 0$. Hence $I_{\infty_j} \neq 0$. For general pair of $(\kappa_{1j},\kappa_{2j})$, we can see the coefficient of s_1^a is not zero by the same way, after acting $(\partial/\partial s_2)^b$ on I_{∞_j}.
(The calculation of $I_\mathfrak{p}$ at good $\mathfrak{p}$): By the mapping $M_2(k_\mathfrak{p}) \ni x \to \alpha(\mathfrak{p}^\delta)x\alpha(\mathfrak{p}^{-\delta}) \in M_2(k_\mathfrak{p})$, we can assume the conductor $-\delta$ of ψ is 0. By the Iwasawa decomposition for GL_2, we can take the complete system of representatives

$$Z_0(k_\mathfrak{p})\backslash H^1(k_\mathfrak{p})/\mathbb{U}_{0,0} = \bigsqcup_{2r+m=n,\,x\in k_\mathfrak{p}}(\varpi^r n(x)\alpha(\varpi^m),\alpha(\varpi^n))$$

where ϖ is the uniformaizer of $\mathfrak{p}$ and $\mathbb{U}_{0,0}$ is defined in (**??**). We calculate

$$\rho(\varpi^r n(x)\alpha(\varpi^m),\alpha(\varpi^n))(v_{-1},\alpha_{-1}) = \left(\begin{bmatrix} 0 & \varpi^{-m-r} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \varpi^{n-m-r} & \varpi^{-m-r}x \\ 0 & \varpi^{-r} \end{bmatrix}\right)$$

with $2r + m = n$. So, $r = 0$, if both of ϖ^{-r} and $\varpi^{n-m-r} = \varpi^r$ is in $\mathfrak{o}$. By lemma **??**, $W_{j\mathfrak{p}}(\alpha(\varpi^m)) = 0$ for $m < 0$. Both of $r(1,h)\varphi_\mathfrak{p}(v_{-1},1)$ and $W_{1v}(h_1)W_{2v}(h_2)$ is not zero, iff $(h_1,h_2) \in Z_0(k_\mathfrak{p})(n_x,1)\mathbb{U}_{0,0}$ with $x \in \mathfrak{o}_\mathfrak{p}$. Hence $I_\mathfrak{p} = W_{1\mathfrak{p}}(1)W_{2\mathfrak{p}}(1) \neq 0$.
(The calculation of $I_\mathfrak{p}$ at bad $\mathfrak{p}$): We treat only the case of $\mathfrak{e}_1 \geq \mathfrak{e}_2$ (then $\mathfrak{e}_1 > 0$ automoatically), since the case of $\mathfrak{e}_1 < \mathfrak{e}_2$ can be treated symmetrically. By our definition, $\varphi_\mathfrak{p}$ is stable on $\mathbb{U}_{\mathfrak{e}_1,\mathfrak{e}_2}$ in the case of $\eta_\mathfrak{p} = 1$ (resp. semi-stable in the csae of $\eta_\mathfrak{p} \neq 1$). Consider the representatives such as

$$\alpha(\varpi^{\mathfrak{e}_i})GL_2(\mathfrak{o})\alpha(\varpi^{-\mathfrak{e}_i})/\Gamma_0(\mathfrak{e}_i) = \{n(j)\}_{j\in\mathfrak{p}^{-\mathfrak{e}_i}/\mathfrak{o}} \sqcup \{\varpi^{-\mathfrak{e}_i}w(\varpi^{2\mathfrak{e}_i})n(j)\}_{j\in\mathfrak{p}^{-\mathfrak{e}_i+1}/\mathfrak{o}}.$$

Using the Iwasawa decomposition for GL_2 with respect to the compact maximal subgroup $\alpha(\varpi^{\mathfrak{e}_i})GL_2(\mathfrak{o})\alpha(\varpi^{-\mathfrak{e}_i})$, we can find at once that all the elements of $Z_0(k_\mathfrak{p})\backslash H^1(k_\mathfrak{p})/\mathbb{U}_{\mathfrak{e}_1,\mathfrak{e}_2}$ to be considered are reduced to the following four types.

i)-type: $(\varpi^r n(x)\alpha(\varpi^m), \alpha(\varpi^n))$ with $m + 2r = n$ and $m, n \geq 0$;
ii)-type: $(\varpi^r n(x)\alpha(\varpi^m) w(\varpi^{\mathfrak{e}_1}) n(s), \alpha(\varpi^n))$ with $2r + m + \mathfrak{e}_1 - n \geq 0$ and $s \in \mathfrak{p}^{-\mathfrak{e}_1-1}$ modulo $\mathfrak{o}_\mathfrak{p}$;
iii)-type: $(\varpi^r n(x)\alpha(\varpi^m), \alpha(\varpi^n) w(\varpi^{\mathfrak{e}_2}) n(t))$ with $m - n + \mathfrak{e}_2 - 2r \geq 0$ and $t \in \mathfrak{p}^{-\mathfrak{e}_2-1}$ modulo $\mathfrak{o}_\mathfrak{p}$;
iv)-type: $(\varpi^r n(x)\alpha(\varpi^m) w(\varpi^{\mathfrak{e}_1}) n(s), \alpha(\varpi^n) w(\varpi^{\mathfrak{e}_2}) n(t))$ with $2r + m + \mathfrak{e}_1 - n + \mathfrak{e}_2$, and $s \in \mathfrak{p}^{-\mathfrak{e}_1-1}$ modulo $\mathfrak{o}_\mathfrak{p}$, $t \in \mathfrak{p}^{-\mathfrak{e}_2-1}$ modulo $\mathfrak{o}_\mathfrak{p}$.

In the case of $\mathfrak{e}_1 = 0$ (resp. $\mathfrak{e}_2 = 0$), it is sufficient to consider i) and iii)-types (resp. i) and ii)-types). Let us calculate the contributions to $I_\mathfrak{p}$ of each types.
i)-type: If we require

$$\rho(h)\alpha_{-1} = \begin{bmatrix} \varpi^r & \varpi^{-r-m}x \\ 0 & \varpi^{-r} \end{bmatrix} \in M_2(\mathfrak{o}_\mathfrak{p})$$

then $r = 0$, so $m - n \geq 0$. Further if we require

$$\rho(h)v_{-1} = \begin{bmatrix} 0 & \varpi^{-m-r} \\ 0 & 0 \end{bmatrix} \in \begin{bmatrix} \mathfrak{p}^{\mathfrak{e}_2} & \mathfrak{o} \\ \mathfrak{p}^{\mathfrak{e}} & \mathfrak{p}^{\mathfrak{e}_2} \end{bmatrix},$$

then $m = 0$, and $n = 0$. So, $x \in \mathfrak{o}_\mathfrak{p}$ is needed. Thus, the total contribution of i)-type is $W_{1\mathfrak{p}}(1)W_{2\mathfrak{p}}(1)$.
ii)-type: We can assume $\mathfrak{e}_1 > 0$, as far as we consider this case. Suppose

$$\rho(h)(v_{-1}, \alpha_{-1}) = \left(\begin{bmatrix} 0 & s\varpi^{-m-r} \\ 0 & \varpi^{-m-r} \end{bmatrix}, \begin{bmatrix} s\varpi^{\mathfrak{e}_1+r} & \varpi^{-r-\mathfrak{e}_1} - s\varpi^{-m-r}x \\ \varpi^{\mathfrak{e}_1+r} & \varpi^{-r-m}x \end{bmatrix}\right) \in \begin{bmatrix} \mathfrak{p}^{\mathfrak{e}_2} & \mathfrak{o} \\ \mathfrak{p}^{\mathfrak{e}} & \mathfrak{p}^{\mathfrak{e}_2} \end{bmatrix} \oplus M_2(\mathfrak{o}_\mathfrak{p}).$$

Then $m + r \leq -\mathfrak{e}_1 < 0$ and $ord_\mathfrak{p}(s) - m - r \geq 0$. Note that $ord_\mathfrak{p}(s) - m - r \geq -\mathfrak{e}_1 + 1 - m - r > -\mathfrak{e}_1 - m - r \geq 0$. Hence $\varpi^{-r-m}\delta \in \mathfrak{o}_\mathfrak{p}$ for $y \in \mathfrak{p}^{-1}$, and $\varphi_\mathfrak{p}(\rho(n_y h_1, h_2)(v_{-1}, a_{-1})) = \varphi_\mathfrak{p}(\rho(h_1, h_2)(v_{-1}, a_{-1}))$. However we can see the contribution cancels, by using

$$W_{1\mathfrak{p}}(n_y h_1) = \psi_\mathfrak{p}(y) W_{1\mathfrak{p}}(h_1). \tag{1.16}$$

iii)-type: We can assume $\mathfrak{e}_2 > 0$, as far as we consider this case. We first going to see that, in the case of $t \neq 0$, this type has no contribution. Suppose that

$$\rho(h)(v_{-1}, \alpha_{-1}) = \left(\begin{bmatrix} -\varpi^{\mathfrak{e}_2-m-r} & t\varpi^{\mathfrak{e}_2-m-r} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \varpi^{\mathfrak{e}_2-m-r}x & \varpi^{n-m-r} - t\varpi^{\mathfrak{e}_2-m-r}x \\ \varpi^{\mathfrak{e}_2-r} & t\varpi^{\mathfrak{e}_2-r} \end{bmatrix}\right) \in \begin{bmatrix} \mathfrak{p}^{\mathfrak{e}_2} & \mathfrak{o} \\ \mathfrak{p}^{\mathfrak{e}} & \mathfrak{p}^{\mathfrak{e}_2} \end{bmatrix} \oplus M_2(\mathfrak{o}_\mathfrak{p}).$$

Then $-m - r \geq 0$ and $t\varpi^{\mathfrak{e}_2-m-r} \in \mathfrak{o}_\mathfrak{p}$. If $t\varpi^{\mathfrak{e}_2-m-r} \in \mathfrak{p}$, then we can conclude the contribution cancels by using (**??**). So, we can assume $t\varpi^{\mathfrak{e}_2-m-r} \in \mathfrak{o}_\mathfrak{p}^\times$. Further, if $m > 0$, then $t\varpi^{\mathfrak{e}_2-r} \in \mathfrak{p}$ and $\det(\rho(h)\alpha_{-1}) = (-\varpi^{\mathfrak{e}_2-m-r}x)(t\varpi^{\mathfrak{e}_2-r}) - (\varpi^{\mathfrak{e}_2-r})(-\varpi^{-m+n-r} - t\varpi^{\mathfrak{e}_2-m-r}x) \in \mathfrak{p}$, which is a contradiction. So we conclude that $m = 0$ and $ord_\mathfrak{p}(t) - r + \mathfrak{e}_2 = 0$. However, as noted before $0 \leq -m - r = -r = -(ord_\mathfrak{p}(t) + \mathfrak{e}_2)$, which is less than zero since $t \in \mathfrak{p}^{-\mathfrak{e}_2-1}/\mathfrak{o}_\mathfrak{p}$. This is a contradiction. In the case of $t = 0$, since $\mathfrak{e}_2 - m - r \geq \mathfrak{e}_2 > 0$, the contribution is also founded to be zero by using (**??**).
iv)-type: We can assume $\mathfrak{e}_1, \mathfrak{e}_2 > 0$ and will see the contribution of this type is zero. Suppose that

$$\rho(h)v_{-1} = \begin{bmatrix} s\varpi^{\mathfrak{e}_2-m-r} & st\varpi^{\mathfrak{e}_2-m-r} \\ \varpi^{\mathfrak{e}_2-m-r} & t\varpi^{\mathfrak{e}_2-m-r} \end{bmatrix} \in \begin{bmatrix} \mathfrak{p}^{\mathfrak{e}_2} & \mathfrak{o} \\ \mathfrak{p}^{\mathfrak{e}} & \mathfrak{p}^{\mathfrak{e}_2} \end{bmatrix},$$
$$\rho(h)\alpha_{-1} = \begin{bmatrix} \varpi^{\mathfrak{e}_2+r+\mathfrak{e}_1} - sx\varpi^{\mathfrak{e}_2-m-r} & t\varpi^{\mathfrak{e}_2+\mathfrak{e}_1+r} - s\varpi^{n-m-r} - st\varpi^{\mathfrak{e}_2-m-r}x \\ -\varpi^{\mathfrak{e}_2-m-r}x & \varpi^{n-m-r} - t\varpi^{\mathfrak{e}_2-m-r}x \end{bmatrix} \in M_2(\mathfrak{o}_\mathfrak{p}).$$

To begin with, we notice $st\varpi^{\mathfrak{e}_2-m-r} \in \mathfrak{o}$. Assume $st \neq 0$. If $st\varpi^{\mathfrak{e}_2-m-r} \in \mathfrak{p}$, the contribution cancels by (**??**). So, we can assume $st\varpi^{\mathfrak{e}_2-m-r} \in \mathfrak{o}_\mathfrak{p}^\times$. However, under this assumption, $t\varpi^{-\mathfrak{e}_1-m-r} =$

$s^{-1}\varpi^{-\mathfrak{e}}(st\varpi^{\mathfrak{e}_2-m-r})$ should be in $\mathfrak{o}$, which is impossible since $s \in \mathfrak{p}^{-\mathfrak{e}_1-1}$. Consequently, $st = 0$. Suppose $s = 0$. Since $t\varpi^{\mathfrak{e}_2-m-r} \in \mathfrak{p}^{\mathfrak{e}_1}$, $ord_{\mathfrak{p}}(t) + \mathfrak{e}_2 - m - r \geq \mathfrak{e}_1 > 0$ and the contribution cancels by (**??**). Suppose $t = 0, s \neq 0$. Then $-\mathfrak{e}_1 - m - r \geq 0$ from $-\varpi^{\mathfrak{e}_2-m-r} \in \mathfrak{p}^{\mathfrak{e}}$. Hence $ord_{\mathfrak{p}}(s) + \mathfrak{e}_2 - m - r = ord_{\mathfrak{p}}(s) + \mathfrak{e} - \mathfrak{e}_1 - m - r > 0$ so $s\varpi^{\mathfrak{e}_2-m-r} \in \mathfrak{p}$, and thus we see the contribution cancels by (**??**).
Summing up the above calculations, $W_{F,\mathfrak{p}}(1)$ is not zero, and F is also not zero.

Novodvorsky zeta integral: Let $W(\pi_v, \psi_{1,-1})$ be the space of Whitakker functions associated to $\psi_{1,-1}$ of $\pi_v \in \widehat{GSp_2(k_v)}$. The Novodvorsky zeta integral of $W \in W(\pi_{\mathfrak{p}}, \psi_{c_1,c_2})$ twisted by $\mu \in \widehat{k_{\mathfrak{p}}^{\times}}$ is defined by

$$Z_N(s, W, \mu) = \int_{k_{\mathfrak{p}}} \int_{k_{\mathfrak{p}}^{\times}} W(\begin{bmatrix} y & & & \\ & y & & \\ & & 1 & \\ & x & & 1 \end{bmatrix})\mu(y)|y|^{s-\frac{3}{2}} dx d^{\times} y.$$

According to Lemma 4.1.1 of Roberts, Schmidt [**?**], if W is semi-paramodular on $K_{\mathfrak{p}}(\mathfrak{e}, \mathfrak{f})$ as in our sense (c.f Definition **??**), then the Novodvorsky zeta integral is equal to the easier zeta integral

$$Z(s, W, \mu) = \int_{k_{\mathfrak{p}}^{\times}} W(diag[y, y, 1, 1])|y|^{s-\frac{3}{2}}\mu(y) d^{\times} y.$$

The calculation of this zeta integral is similar to that of $W_{F,\mathfrak{p}}(1)$. That is done by replacing $\varphi_{\mathfrak{p}}(x_1, x_2)$ to $\varphi_{\mathfrak{p}}(yx_1, yx_2)$, and the relation of $\det(h_1)$ and $\det(h_2)$ to that of $\det(h_1)y$ and $\det(h_2)$. Then, we have

$$W_{F\mathfrak{p}}(diag[y, y, 1, 1]) = |y| \sum_{l=0}^{ord_{\mathfrak{p}}(y)} W_{1\mathfrak{p}}(\omega^l \alpha(y\varpi^{-l}))W_{2\mathfrak{p}}(\alpha(\varpi^l)),$$

and thus $Z(s, W_{F\mathfrak{p}})$ is calculated as

$$\int_{k_{\mathfrak{p}}^{\times}} \sum_{l=0}^{ord_{\mathfrak{p}}(y)} W_{1\mathfrak{p}}(\omega^l \alpha(y\varpi^{-l}))W_{2\mathfrak{p}}(\alpha(\varpi^l))|y|^{s-\frac{1}{2}} d^{\times} y = \int_{k_{\mathfrak{p}}^{\times}} \sum_{l=0}^{ord_{\mathfrak{p}}(y)} W_{1\mathfrak{p}}(\alpha(y\varpi^{-l}))W_{2\mathfrak{p}}(\alpha(\varpi^l))\eta_{\mathfrak{p}}^{-1}(\omega^l)|y|^{s-\frac{1}{2}} d^{\times} y$$

$$= \int_{k_{\mathfrak{p}}^{\times}} W_{1\mathfrak{p}}(\alpha(x))|x|^{s-\frac{1}{2}} d^{\times} x \int_{k_{\mathfrak{p}}^{\times}} W_{2\mathfrak{p}}(\alpha(z))\eta_{\mathfrak{p}}^{-1}(z)|z|^{s-\frac{1}{2}} d^{\times} z = L(s, \tilde{\sigma}_{1\mathfrak{p}})L(s, \tilde{\sigma}_{2\mathfrak{p}}).$$

γ-factor: It is known that there exists γ-factor $\gamma(s, \pi, \mu, \psi_{1,-1})$ satisfying the functional equation for every $W \in W(\pi_{\mathfrak{p}}, \psi_{1,-1})$:

$$Z_N(1 - s, \varrho([\ _{w_1}\ ^{w_1}\ ])W, \eta_{\mathfrak{p}}\mu^{-1}) = \gamma(s, \pi, \mu, \psi_{1,-1})Z_N(s, W, \mu), \tag{1.17}$$

where ϱ means the action by the right translation. We calculate

$$\varrho([\ _{w_1}\ ^{-w_1}\ ])W_{F\mathfrak{p}}(diag[y, y, 1, 1]) = |y| \sum_{m=0}^{l+\mathfrak{e}} W_{1\mathfrak{p}}(\omega^{l+\mathfrak{e}_2-m}\alpha(\omega^m)w(\varpi^{\mathfrak{e}_1}))W_{2\mathfrak{p}}(\alpha(\omega^{l+\mathfrak{e}-m})w(\varpi^{\mathfrak{e}_2}))$$

with $l = ord_{\mathfrak{p}}(y)$. Hence the left hand side of (??) is calculated as

$$\begin{aligned}
&\int_{k_{\mathfrak{p}}^\times} \eta_{\mathfrak{p}}(y) \sum_{m=0}^{l+\epsilon} W_{1\mathfrak{p}}(\varpi^{l+\epsilon_2-m} a(\varpi^m) w(\varpi^{\epsilon_1})) W_{2\mathfrak{p}}(a(\varpi^{l+\epsilon-m}) w(\varpi^{\epsilon_2})) |y|^{1+(1-s)-\frac{3}{2}} d^\times y \\
= &\int_{k_{\mathfrak{p}}^\times} \eta_{\mathfrak{p}}(y) \eta_{\mathfrak{p}}(\varpi^{-l+m-\epsilon_2}) \sum_{m=0}^{l+\epsilon} W_{1\mathfrak{p}}(a(\varpi^m) w(\varpi^{\epsilon_1})) W_{2\mathfrak{p}}(a(\varpi^{l+\epsilon-m}) w(\varpi^{\epsilon_2})) |y|^{(1-s)-\frac{1}{2}} d^\times y \\
= &\int_{k_{\mathfrak{p}}^\times} W_{1\mathfrak{p}}(a(x) w(\varpi^{\epsilon_1})) \eta_{\mathfrak{p}}(x) |x|^{(1-s)-\frac{1}{2}} d^\times x \int_{k_{\mathfrak{p}}^\times} \eta_{\mathfrak{p}}(\varpi^{-\epsilon_2}) W_{2\mathfrak{p}}(a(\varpi^{\epsilon}) a(z) w(\varpi^{\epsilon_2})) |z|^{(1-s)-\frac{1}{2}} d^\times z \\
= &\int_{k_{\mathfrak{p}}^\times} W_{1\mathfrak{p}}(a_x w_1) q^{\epsilon_1 s} \eta_{\mathfrak{p}}(x) |x|^{(1-s)-\frac{1}{2}} d^\times x \int_{k_{\mathfrak{p}}^\times} W_{2\mathfrak{p}}(a_z w_1) q^{-\epsilon_1 s} |z|^{(1-s)-\frac{1}{2}} d^\times z \\
= &\int_{k_{\mathfrak{p}}^\times} \varrho(w_1) W_{1\mathfrak{p}}(a_x) \eta_{\mathfrak{p}}(x) |x|^{(1-s)-\frac{1}{2}} d^\times x \int_{k_{\mathfrak{p}}^\times} \varrho(w_1) W_{2\mathfrak{p}}(a_z) |z|^{(1-s)-\frac{1}{2}} d^\times z
\end{aligned}$$

which is equal to $Z(1-s, \varrho(w_1)W_1, \eta_{\mathfrak{p}}) Z(1-s, \varrho(w_1)W_2)$. Thus, noting $(W_2 \otimes \eta^{-1}) \otimes \eta \in \tilde{\tilde{\sigma}}_2 = \sigma_2$,

$$\gamma(s, \pi, \psi_{1,-1}) = \gamma(s, \tilde{\sigma}_1, \psi) \gamma(s, \tilde{\sigma}_2, \psi). \tag{1.18}$$

L-factor, ϵ-factor: At finite place $\mathfrak{p}$, the $\mathbb{C}[q^s, q^{-s}]$ module generated by the $Z(s, W, \mu)$ for $W \in W(\pi_{\mathfrak{p}}, \psi_{1,-1})$ is a principal fractional ideal of $\mathbb{C}(q^{-s})$. The genarator of the ideal is in the form $Q(q^{-s})^{-1}$ with $Q(X) \in \mathbb{C}[X]$ such that $Q(0) = 1$ (c.f. [?]). The $L(s, \pi_{\mathfrak{p}}, \mu)$ is defined by $Q(q^{-s})^{-1}$ and the ϵ-factor is defined by

$$\epsilon(s, \pi_{\mathfrak{p}}, \mu, \psi_{1,-1}) = \gamma(s, \pi_{\mathfrak{p}}, \mu, \psi_{1,-1}) \frac{L(s, \pi_{\mathfrak{p}}, \mu)}{L(1-s, \tilde{\pi}_{\mathfrak{p}}, \mu^{-1})} \tag{1.19}$$

PROPOSITION 1.5 *The irreducible $\pi_{\mathfrak{p}}$ associated to $\theta(g; f_1 \boxtimes f_2)$ has, for every $\mu \in \widehat{k_{\mathfrak{p}}^\times}$,*

$$L(s, \pi_{\mathfrak{p}}, \mu) = L(s, \tilde{\sigma}_{1\mathfrak{p}} \otimes \mu) L(s, \tilde{\sigma}_{2\mathfrak{p}} \otimes \mu), \quad \epsilon(s, \pi_{\mathfrak{p}}, \mu, \psi_{1,-1}) = \epsilon(s, \tilde{\sigma}_{1\mathfrak{p}} \otimes \mu, \psi) \epsilon(s, \tilde{\sigma}_{2\mathfrak{p}} \otimes \mu, \psi).$$

PROOF. It suffices to show in the case of $\mu = 1$. To begin with, we notice that, if $Q(X) = \prod_{i=1}^n (1 - \alpha_i X)$ represents the polynomial of $L(s, \pi_{\mathfrak{p}})$, then $\tilde{Q}(X) = \prod_{i=1}^n (1 - \alpha_i^{-1} X)$ does of $L(s, \tilde{\pi}_{\mathfrak{p}})$. Assume that $L(s, \tilde{\sigma}_{1\mathfrak{p}}) L(s, \tilde{\sigma}_{2\mathfrak{p}})$ is not $L(s, \pi_{\mathfrak{p}})$. Then, we can write $L(s, \tilde{\sigma}_{1\mathfrak{p}}) L(s, \tilde{\sigma}_{2\mathfrak{p}}) = L(s, \pi_{\mathfrak{p}}) r(q^{-s})$ by a certain polynomial $r(X) = \prod_{j=1}^m (1 - \beta_j X) \in \mathbb{C}[X]$, since $L(s, \pi_{\mathfrak{p}})$ is the generator of the principal ideal. So,

$$\epsilon(s, \tilde{\sigma}_{1\mathfrak{p}}, \psi) \epsilon(s, \tilde{\sigma}_{2\mathfrak{p}}, \psi) = \gamma(s, \tilde{\sigma}_{1\mathfrak{p}}, \psi) \gamma(s, \tilde{\sigma}_{2\mathfrak{p}}, \psi) \frac{L(s, \tilde{\sigma}_{1\mathfrak{p}}) L(s, \tilde{\sigma}_{2\mathfrak{p}})}{L(1-s, \sigma_{1\mathfrak{p}}) L(1-s, \sigma_{2\mathfrak{p}})},$$

as well as (??). Hence, by (??), we have

$$\frac{\epsilon(s, \pi_{\mathfrak{p}}, \psi_{1,-1})}{\epsilon(s, \tilde{\sigma}_{1\mathfrak{p}}, \psi) \epsilon(s, \tilde{\sigma}_{2\mathfrak{p}}, \psi)} = \frac{\prod_{j=1}^m (1 - \beta_j^{-1} q^{s-1})}{\prod_{j=1}^m (1 - \beta_j q^{-s})}.$$

However, the right hand side is a monomial of q^{-s} since all the ϵ-factors in the left hand side are so. Then $\{\beta_j\}_{j=1}^m$ should coincide with $\{q\beta_j\}_{j=1}^m$ as a set, which is impossible. This completes the proof. □

According to Roberts, Schmidt [?], if the central character η_π of $\pi = W(\pi, \psi)$ is trivial, the ε-factor is written as $\varepsilon_\pi q^{-N_\pi(s-\frac{1}{2})}$ for the eigenvalue $\varepsilon_\pi \in \{\pm 1\}$ of the newvector $W_{\mathfrak{p}}$ with respect to the Atkin-Lehner type operator, the right translation by the element

$$u_{N_{\pi_{\mathfrak{p}}}} = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & \varpi^{N_{\pi_{\mathfrak{p}}}} & & \\ \varpi^{N_{\pi_{\mathfrak{p}}}} & & & \end{bmatrix}.$$

For $F = \theta(g; f_1 \boxtimes f_2)$, in the case of $\eta_{\sigma 1} = \eta_{\sigma 2} = 1$, we find $W_{F\mathfrak{p}}$ at every finite $\mathfrak{p}$ is just the local newvector by the coincidence of $N_{\pi_{\mathfrak{p}}}$. Also by the Weil representation, we can see directly $\varepsilon_\pi = \varepsilon_1 \varepsilon_2$, where $\varepsilon_i \in \{\pm 1\}$ is the eigenvalue of W_i for the Atkin-Lehner operator.

semi-paramodular conjecture: As a generalization and globalization of [?], we naturally guess the followings also for the nontrivial character case by our evidence. At finite place $\mathfrak{p}$, for positive integer n, put

$$W_n(\pi_{\mathfrak{p}}, \psi_{1,-1}) = \{W \in W(\pi_{\mathfrak{p}}, \psi_{1,-1}) \mid W(gu) = \eta_\pi(d) W(g), \text{ for every } u \in K_{\mathfrak{p}}(n, c(\eta_\pi)) \text{ such as}$$
$$u = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & d & * \\ * & * & * & * \end{bmatrix} \quad (\text{mod } \mathfrak{p}^n)\}.$$

Conjecture 2 *Suppose that an irreducible generic cuspidal representation* $\pi = \otimes_v \pi_v \in \Pi(GSp_2(\mathbb{A}))$ *is unitary, and all* π_v *are infinite dimensional. Then,* $\varepsilon(s, \pi) := \prod_v \varepsilon(s, \pi_v, \psi_{1,-1})$ *is written as* $\varepsilon_\pi |D_{k/\mathbb{Q}}|^{2-4s} N_{k/\mathbb{Q}}(c_\pi)^{\frac{1}{2}-s}$ *for* $\varepsilon_\pi = \varepsilon(\frac{1}{2}, \pi)$ *with* $|\varepsilon_\pi| = 1$. *Here* $c_\pi = \prod_{\mathfrak{p}} \mathfrak{p}^{N_{\pi_{\mathfrak{p}}}}$ *is characterized by the condition* $\dim_{\mathbb{C}} W_{N_\pi}(\pi_{\mathfrak{p}}, \psi_{1,-1}) = 1$. *In this case,* $W \in W_{N_\pi}(\pi_{\mathfrak{p}}, \psi_{1,-1})$ *satsifies* $Z_N(s, W) = L(s, \pi_{\mathfrak{p}})$, *up to constant multiple.*

cuspidality: Put the Siegel (resp. Klingen) parabolic subgroup P (resp. Q) of Sp_2 as

$$P = N_P M_P = \{\begin{bmatrix} 1_2 & S \\ & 1_2 \end{bmatrix} \mid S = {}^tS \in M(2)\}\{\begin{bmatrix} g & \\ & {}^tg^{-1} \end{bmatrix} \mid g \in GL(2)\},$$
$$Q = N_Q M_Q = \left\{\begin{bmatrix} 1 & & & * \\ * & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{bmatrix}\right\}\left\{\begin{bmatrix} a & & b & \\ & t & & \\ c & & d & \\ & & & t^{-1} \end{bmatrix} \mid \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2), t \in GL(1)\right\}.$$

Define the two embeddings e_P, e_Q of $GL(2)$ into $GSp(2)$ by

$$e_P(g) = \begin{bmatrix} g & \\ & {}^tg^{-1} \end{bmatrix}; \quad e_Q(g) = \begin{bmatrix} a_g & & b_g & \\ & 1 & & \\ c_g & & d_g & \\ & & & \det(g) \end{bmatrix}.$$

If $E \in \mathcal{A}(GSp_2(\mathbb{A}))$ is a non-cuspform, by the definition of cuspform, we obtain automorphic forms of $GL_2(\mathbb{A})$ by the integrals

$$\Phi_P(E)(g; h) = \int_{N_P(k)\backslash N_P(\mathbb{A})} E(n e_P(g) h) dn, \text{ or } \Phi_Q(E)(g; h) = \int_{N_Q(k)\backslash N_Q(\mathbb{A})} E(n e_Q(g) h) dn$$

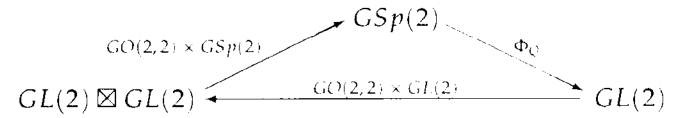
for some $h \in GSp_2(\mathbb{A})$ where dn in each integrals are suitable Haar measures on $N_P(\mathbb{A}), N_Q(\mathbb{A})$. Let us observe the cuspidality of $F = \theta(g; f_1 \boxtimes f_2)$. Consider the case of $\sigma_1 = \sigma_2$. In this case, F is non-cuspidal since $L_S^{st}(s, F) = \zeta_S(s, k) L_S(s, \tilde{\sigma}_1 \times \sigma_1)$ for the set S of bad places has a double pole at $s = 1$. But cuspforms cannot such property according to Kudla-Rallis [?]. We can also check the noncuspidality by $\Phi_Q(F;g_1) \neq 0$ at

$$g_1 = \prod_{\mathfrak{p}} \begin{bmatrix} \varpi_{\mathfrak{p}}^{e_i} & \varpi_{\mathfrak{p}}^{e_i} & & \\ & 1 & & \\ & & \varpi_{\mathfrak{p}}^{e_i} & \\ & & 1 & 1 \end{bmatrix}.$$

We can regard $\Phi_Q(F;g_1)$ as a theta lift from $GO(2,2)$ to $GL(2)$, that is,

$$\theta(g;f_1 \boxtimes f_2, \varphi_1) = \int_{\mathbb{A}^\times H^1(k)\backslash H^1(\mathbb{A})} \sum_{x \in M_2(k)} (r(g,(h_1,h_2))\varphi^1)(x) f_1(h_1) f_2(h_2) dh,$$

where $\varphi_{\mathfrak{p}}^1(x) = \mathrm{ch}(\begin{bmatrix} \mathfrak{o} & \mathfrak{p}^{-\delta} \\ \mathfrak{p}^{e_i-\delta} & \mathfrak{o} \end{bmatrix})$ in the case of $c(\eta_{\mathfrak{p}}) = 0$, and $\eta_{\mathfrak{p}}(\varpi^{\delta} b_x)\mathrm{ch}(\begin{bmatrix} \mathfrak{o} & \mathfrak{p}^{-\delta}\mathfrak{o}^\times \\ \mathfrak{p}^{e_i-\delta} & \mathfrak{o} \end{bmatrix})$ otherwise (ch means characteristic function). It is easy to see this theta lift is not trivial by noting $\int_{GL_2(k)\backslash GL_2(\mathbb{A})} f_1(h_1) f_2(h_1) dh_1 \neq 0$. This means the rotation:



Next, consider the case that σ_1 is corresponding to some Eisenstein series, i.e., all κ_{1j} is equal to κ for some $\kappa \geq 1$ and $L(s,\sigma_1) = L(s - \frac{\kappa-1}{2}, \chi) L(s + \frac{\kappa-1}{2}, \chi^{-1}\eta)$ for some $\chi \in \mathfrak{X}(k^\times \backslash \mathbb{A}^\times)$. Then the χ-twist of spinor L-function of F is equal to

$$\zeta(s - \frac{\kappa-1}{2}; k) L(s + \frac{\kappa-1}{2}, \eta^{-1}\chi^2) L(s, \tilde{\sigma}_2 \otimes \chi)$$

where $\zeta(s;k)$ is the Dedekind zeta function of k. In particular, it has a simple pole at $s = \frac{\kappa+1}{2}$ by the final Remark of Jacquet-Shalika [?]. Moriyama [?] showed that every spinor L-function of cuspidal generic representation is entire, hence F is not cuspidal. However, we can also know the noncuspidality by finding that the degenerated Whittaker function $W_F^{1,0}(1)$ is not zero. All the elements which may contribute to the Fourier coefficient at 0 are in the $\rho(H(k))$-orbits of i) (v_1, xv_1) with $x \in k$, ii) $(v_1, \begin{bmatrix} & \\ & 1 \end{bmatrix})$, or iii) $(v_1, \begin{bmatrix} 1 & \\ & \end{bmatrix})$. However, one can see easily that the orbits of i) have no contribution, by noting

$$\int_{\mathbb{A}^\times} f_1(\alpha_t) f_2(\alpha_t) d^\times t = \int_{\mathbb{A}^\times} \sum_x W_1(\alpha_{xt}) \sum_y W_2(\alpha_{yt}) d^\times t = 0.$$

For the orbits of ii), by the similar calculation of $W_F^N(1)$, with noting

$$\{h \in H^1(\mathbb{A}) \mid \rho(h)v_1 = v_1, \rho(h)\begin{bmatrix} & \\ & 1 \end{bmatrix} \in \begin{bmatrix} & \\ & 1 \end{bmatrix} + \mathbb{A}v_1\}/\mathbb{A}^\times = \{(n_x, n_{x'}) \mid x, x' \in \mathbb{A}\},$$

$$\{h \in H^1(\mathbb{A}) \mid \rho(h)(v_1, \begin{bmatrix} & \\ & 1 \end{bmatrix}) = (v_1, \begin{bmatrix} & \\ & 1 \end{bmatrix})\}/\mathbb{A}^\times = \{(n_x, 1) \mid x \in \mathbb{A}\},$$

one can see easily that the contribution is reduced to the value of $c_{f_2} W_{f_1}(w_1)$, where c_{f_2} is the constant term of f_2. However since $c_{f_2} = 0$, these orbits have no contribution. For the orbits of iii), by the discussion symmetric to that for the orbits of ii), its contribution is reduced to the value of $c_{f_1} W_{f_2}(1)$, which is not zero. Hence $W_F^{1,0}(1)$ is not zero. This means $\Phi_P(F;1) = f_1(g)$, and the rotation below.

$$
\begin{array}{ccccc}
 & & GSp(2) & & \\
 & \nearrow^{GO(2,2)\times GSp(2)} & & \searrow^{\Phi_P} & \\
GL(2)\boxtimes GL(2) & & \xleftarrow{\quad ? \quad} & & GL(2)
\end{array}
$$

Remark that the correspondence $GL(2) \boxtimes GL(2) \longleftrightarrow GL(2)$ is different from the theta correspondence as in the previous case (so we denote it by the question mark '?'). Other than the above two cases, we find $\theta(g; f_1 \boxtimes f_2)$ is cuspidal by theorem 1.8 of [?], which characterizes cuspidality of automorphic representation of $GSp_2(\mathbb{A})$ by standard L-function. The theorem is obtained by observing the L-function of noncuspform and complementing the result of Kudla, Rallis [?]. Summing up the aboves,

THEOREM 1.6 *Suppose $\sigma_1, \sigma_2 \in \Pi(GL_2(\mathbb{A}))$ satisfies the condition at the beginning of this section. Take the newforms $f_1 \in \tilde{\sigma}_1, f_2 \in \sigma_2$ as in (??) and define $F = \theta(g; f_1 \boxtimes f_2)$ by (??). Then,*

i) If one of σ_1 and σ_2 is not cuspidal, then F is not cuspidal. This lifting can be considered as an analogue of Langlands lift along the Siegel parabolic subgroup N_P.

ii) If $\sigma_1 = \sigma_2$, then F is not cuspidal. This lifting can be considered as an analogue of Langlands lift along the Klingen parabolic subgroup N_Q.

iii) Othewise F is cuspidal.

At finite place $\mathfrak{p}$, let $\mathfrak{c}$ be the sum of the conductors of σ_1, σ_2, and $\mathfrak{f}$ be the conductor of η. If $\mathfrak{f} = 0$, then F is 'stable on $K_{\mathfrak{p}}(\mathfrak{c})$'. If $\mathfrak{f} > 0$, then F is $\eta_{\mathfrak{p}}^{-1}$-semi stable on $K_{\mathfrak{p}}(\mathfrak{c}, \mathfrak{f})$ in the sence of Definition ??. The Whittaker function $W_F^{1,-1}(1)$ associated to $\psi_{1,-1}$ is not zero, and it holds

$$Z_N(s, W_{Fv}) = L(s, \pi_v) = L(s, \tilde{\sigma}_{1v})L(s, \tilde{\sigma}_{2v}), \quad \varepsilon(s, \pi_v, \psi_{1,-1}) = \varepsilon(s, \tilde{\sigma}_{1v}, \psi)\varepsilon(s, \tilde{\sigma}_{2v}, \psi)$$

at every plcae v, where $Z_N(s, W_{Fv})$ is the Novodvorsky integral and $\pi = \otimes_v \pi_v$ is the irreducible automorphic representation associated to F.

2 $\mathcal{A}(GL_2(L_{\mathbb{A}})^{\circ}) \longrightarrow \mathcal{A}(GSp_2(\mathbb{A})^{\circ})$ for quadratic extension L/k.

Take a squarefree element $a \in \mathfrak{o}_k^{\times}$ and put $\epsilon = \sqrt{a}$. Let $L = k(\epsilon)$ be the quadratic extension of k with ring of integer $\mathfrak{O}$. Let $\chi \in \mathfrak{X}^1(k^{\times} \backslash k_{\mathbb{A}}^{\times})$ be the class character associated to L/k, and c the generator of $Gal(L/k)$. Let

$$V(k) = \{x \in M_2(L) \mid c(x^{\iota}) = -x\} = \{\begin{bmatrix} a_x & b_x \\ c_x & a_x^c \end{bmatrix} \mid b_x, c_x \in k\}$$

with quadratic form $\langle x, y \rangle = -\mathrm{tr}(xy^{\iota})$. In this section, we write

$$H(k) = k^{\times} \times GL_2(L), \quad H^1(k) = \{(t, h) \in H(k) \mid t^2 = N_{L/k}(\det h)\}$$

and define the action of $H(k)$ (resp. $H(\mathbb{A})$) on $V(k)$ (resp. $V(\mathbb{A})$) by

$$\rho(t, h)x = t^{-1}\alpha(h^{\iota})xh.$$

We require for $\sigma = \otimes_v \sigma_v \in \Pi(GL_2(L_{\mathbb{A}}))$ the following assumptions.

i) σ is cuspidal.

ii) the central character $\eta = \eta_\sigma$ is written as $\mu \circ N_{L/k}$ by a certain $\mu \in \mathfrak{X}^1(k^\times \backslash \mathbb{A}^\times)$.

iii) if the j-th archimedean place ∞_j of k splits into (real) ∞_{j1}, ∞_{j2} in L, then both of $\sigma_{1\infty_j}, \sigma_{2\infty_j}$ are discrete series representation with an identical lowest weight $\kappa_j + 1 (\geq 1)$.

iv) in the case of $L_{\infty_j} \simeq \mathbb{C}$, σ_{∞_j} has Langlands parameter $z \longmapsto diag[(\frac{z}{\sqrt{\bar{z}}})^{\kappa_j}, (\frac{\bar{z}}{\sqrt{z}})^{\kappa_j}]$ with $\kappa_j \in \mathbb{Z}_{>0}$. (Hence, η_{∞_j} is trivial and σ_{∞_j} contains the $2\kappa_j$-th symmetric representation of $SU_2(\mathbb{C})$). And $\mu_{\infty_j}(-1) = (-1)^{\kappa_j}$.

REMARK 2.1 *Even if $\mu_{\infty_{j_i}}(-1) = (-1)^{\kappa_{j_i}+1}$ at some ∞_{j_i} where $L_{\infty_{j_i}} \simeq \mathbb{C}$, we can also consider the theta lift similar to $F = \theta(g; f_\mu)$ below. But, we don't know this theta lift is always non-vanishing. By this lift, we will obtain another F' whose central character is different from that of F. F' may be holomorphic or belong to non-holomorphic discrete series representation at ∞_j (see section and of [?] for the account). If $F' \neq 0$, then it holds $L(s,F)_v = L(s,F')_v$ at all finite v where F' is $GSp_2(\mathfrak{o}_v)$ stable and an eigenform. (Here $L(s,F')_v$ is defined by the Hecke operators explained in the next section at finite v.)*

Automorphic form of $H(\mathbb{A})$: Let $\psi_L = \psi_k \circ Tr_{L/k}$ with ψ_k fixed in the previous section. In the case of $L_{\infty_j} \simeq \mathbb{R} \times \mathbb{R}$, we choose $W_{\sigma\infty_j 1}, W_{\sigma\infty_j 2}$ in the same way as previous section. In the case of $L_{\infty_j} \simeq \mathbb{C}$, by Lemma 13 of Asai [?], we can choose $W_{\sigma\infty_j} \in W(\sigma_{\infty_j}, \psi_{\infty_j})$ so that

$$W_{\sigma\infty_j}(a_y n_x) = (\phi_\alpha(y) \exp(4\pi i \mathrm{Re}(x)))_{-\kappa_j \leq \alpha \leq \kappa_j}, \quad \phi_\alpha(y) = \binom{2\kappa_j}{\kappa_j + \alpha} K_\alpha(4\pi y) y^{\kappa_j + 1}$$

for $y \in \mathbb{R}_{>0}, x \in \mathbb{C}$ and $W_{\sigma\infty_j}(hu) = \mathrm{Sym}_{2\kappa_j}(u) W_{\sigma\infty_j}(hu)$ for $u \in SU_2(\mathbb{C})$. Here K_α is a certain modified Bessel function of order α, which satisfies

$$\int_0^\infty \exp(-at - t^{-1}b) L_\beta^{(\alpha)}(at) t^{\alpha+\beta-1} dt = (-1)^\beta 2(\beta!)^{-1} a^{-\alpha/2} b^{\alpha/2+\beta} K_\alpha(2\sqrt{ab}) \qquad (2.20)$$

for $a, b \in \mathbb{R}_{>0}$, where $L_\beta^{(\alpha)}(x) = \sum_{j=0}^{\beta} \binom{\beta+\alpha}{\beta-\alpha} (-x)^j / j!$ is Laguerre's polynomial (c.f. p.161 of [?]). At finite place $\mathfrak{P}$ of L, we take $W_{\mathfrak{P}} \in W(\sigma_{\mathfrak{P}}, \psi_{\mathfrak{P}})$ so that

$$W_{\mathfrak{P}}(hu) = \eta_{\mathfrak{P}}(a_u) W_{\mathfrak{P}}(h), \quad u \in \Gamma_{0\mathfrak{P}}(\mathfrak{c} + \delta_{L_\mathfrak{P}}, -\delta_{L_\mathfrak{P}})$$

where $\mathfrak{P}^{\delta_{L_\mathfrak{P}}}$ is the local different of $L_\mathfrak{P}$ over $\mathbb{Q}_p$. Set, for $\tilde{h} = (t, h) \in H(\mathbb{A})$,

$$f(h) = \sum_{\alpha \in L^\times} \prod_v W_{\sigma v}(a_\alpha h) \in \mathcal{A}(GL_2(L_\mathbb{A})); \quad f_\mu(\tilde{h}) = \mu^{-1}(t) f(h) \in \mathcal{A}(H(\mathbb{A})). \qquad (2.21)$$

where we set $W_{\sigma v}(a_\alpha h) = W_{\sigma v}(|a_\alpha| h)$ at infinite v. **Schwartz function:** We define the Schwartz function $\varphi_v \in \mathcal{S}(V_v^2)$ corresponding to f_μ as follows.

(At ∞_j where $L_{\infty_j} \simeq \mathbb{R} \times \mathbb{R}$) Define the same Schwartz function φ_{∞_j} for weight $\kappa_j + 1$ as in the previous section. We set φ_j^0 so that the symmetric matrix R_j is a minimal majorant of Q_j assosiated to the quadratic form $\langle \ , \ \rangle$, that is,

$$R_j Q_j^{-1} R_j = Q_j, \ Q_j = \begin{bmatrix} 2 & & & \\ & 2\varepsilon_j^2 & & \\ & & & 1 \\ & & 1 & \end{bmatrix}, \ R_j = \begin{bmatrix} 2 & & & \\ & 2\varepsilon_j^2 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

(At ∞_j where $L_{\infty_j} \sim \mathbb{C}$) Set a homogeneous polynomial $P(x) = (P_i(x))_{1\le i\le 2\kappa_j+1} \in \mathbb{C}[x]$ of degree κ_j so that

$$P(x)^t(u^{2\kappa_j}, u^{2\kappa_j-1}v, \ldots, v^{2\kappa_j}) = (x, \begin{bmatrix} u^2 & uv \\ uv & v^2 \end{bmatrix})^{\kappa_j}$$

for indeterminants u, v. It holds

$$P(xu) = P(x)sym_{2\kappa_j}(u^{-1}), \quad u \in SU_2(\mathbb{C}).$$

We define

$$\varphi_{\infty_j}(x_1, x_2) = P(s_1x_1 + s_2x_2)\mathbf{e}(-\sum_{i=1}^{2}(x_{i(1)}^2 - \varepsilon_j^2 x_{i(2)}^2 + x_{i(3)}^2 + x_{i(4)}^2)),$$

where $\varepsilon_j^2 < 0$ is the j-th conjugate of $\varepsilon^2 \in k$.

(At $\mathfrak{p} = \mathfrak{P}^2$, $c(\mu_\mathfrak{p}) = 0$ case): Different from the previous section, characteristic functions of lattices do not provide images of theta lift which are stable on paramodular groups. For example, $ch(V_\mathfrak{p} \cap M_2(\mathfrak{o}_\mathfrak{p}))^2$ will provide $F \in \mathcal{A}(GSp_2(\mathbb{A}))$ such as

$$F(gu) = \chi_\mathfrak{p}(\det(a_u))F(g), \quad u \in \Gamma_0(\mathfrak{p}^n)$$

with the $\mathfrak{p}$ adic order n of norm of the dual lattice of $V_\mathfrak{p} \cap M_2(\mathfrak{o}_\mathfrak{p})$. So, we define $\varphi_\mathfrak{p}$ in the following way. Define for non negative integer n,

$$\mathbb{U}_n = \mathfrak{o}_\mathfrak{p}^\times \times \{u = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \begin{bmatrix} \mathfrak{O}_\mathfrak{P} & \mathfrak{P}^{-1} \\ \mathfrak{P}^{n+1} & \mathfrak{O}_\mathfrak{P} \end{bmatrix} \mid \det(u) \in \mathfrak{O}_\mathfrak{P}^\times\} \cap H^1(k_\mathfrak{p}).$$

For the simplicity, we assume the local different $\mathfrak{p}^\delta$ of the $k_\mathfrak{p}$ over $\mathbb{Q}_p$ is $\mathfrak{o}_\mathfrak{p}$. To begin with, we set, for $y \in V_\mathfrak{p}$

$$\varphi_\mathfrak{p}^0(y) = \begin{cases} \chi_\mathfrak{p}(\varpi b_y), & \text{if } y \in \begin{bmatrix} \mathfrak{O}_\mathfrak{P} & \varpi^{-1}\mathfrak{o}_\mathfrak{p}^\times \\ \varpi\mathfrak{o}_\mathfrak{p} & \mathfrak{O}_\mathfrak{P} \end{bmatrix}, \\ 0, & \text{otherwise.} \end{cases}$$

Let $r_\mathfrak{p}^1$ be the Weil representation restricted to $SL_2(k_\mathfrak{p})$. If $x \in \mathfrak{o}_\mathfrak{p}, t \in \mathfrak{o}_\mathfrak{p}^\times, y \in \mathfrak{p}$, then clearly it holds

$$r_\mathfrak{p}^1(n_x)\varphi_\mathfrak{p}^0 = r_\mathfrak{p}^1(t^{-1}\alpha(t^2))\varphi_\mathfrak{p}^0 = r_\mathfrak{p}^1({}^tn_y)\varphi_\mathfrak{p}^0 = \varphi_\mathfrak{p}^0. \tag{2.22}$$

PROPOSITION 2.2 *Take the sum*

$$\varphi_\mathfrak{p}^1(y) = \varphi_\mathfrak{p}^0(y) + \sum_{i \in \mathfrak{o}/\mathfrak{p}} r_\mathfrak{p}^1(n_i w_1)\varphi_\mathfrak{p}^0(y). \tag{2.23}$$

Then $\varphi_\mathfrak{p}^1$ is not identically zero. For $u \in SL_2(\mathfrak{o}_\mathfrak{p})$ and $(1, v) \in \mathbb{U}_0$, it holds $r_\mathfrak{p}^1(u)\varphi_\mathfrak{p}^1 = \rho(1,v)\varphi_\mathfrak{p}^1 = \varphi_p^1$.

PROOF. To begin with, we notice

$$\sum_{i \in \mathfrak{o}/\mathfrak{p}} r_\mathfrak{p}^1(n_i w_1)\varphi_\mathfrak{p}^0(y) = \begin{cases} cq^{-\frac{3}{2}}G(\chi_\mathfrak{p}, \psi_\mathfrak{p})\chi_\mathfrak{p}(-c_y), & \text{if } y \in \begin{bmatrix} \mathfrak{P}^{-1} & \varpi^{-1}\mathfrak{o}_\mathfrak{p} \\ \mathfrak{o}_\mathfrak{p}^\times & \mathfrak{P}^{-1} \end{bmatrix} \text{ and } \det(y) \in \mathfrak{o}_\mathfrak{p}, \\ 0, & \text{otherwise.} \end{cases}$$

Here c is the Weil constant, and $G(\chi_\mathfrak{p}, \psi_\mathfrak{p})$ is the Gauß sum associated to $\chi_\mathfrak{p}, \psi_\mathfrak{p}$. Hence $\varphi_\mathfrak{p}^1(n_1) = \varphi_\mathfrak{p}^0(n_1) \neq 0$, the first assertion follows. Since (**??**) is an average sum, it is easy to see $r_\mathfrak{p}^1(u)\varphi_\mathfrak{p}^1 = \varphi_\mathfrak{p}^1$. We see by direect calculation that $\rho(1, n_z)\varphi_\mathfrak{p}^1 = \varphi_\mathfrak{p}^1$ for $z \in \mathfrak{P}^{-1}$ and $\rho(1, {}^tn_z)\varphi_\mathfrak{p}^1 = \varphi_\mathfrak{p}^1$ for $z \in \mathfrak{P}$. This completes the proof. □

Then, for the conductor $\mathfrak{c} = c(\sigma_{\mathfrak{P}})$ of $\sigma_{\mathfrak{P}}$, we define

$$\varphi_{\mathfrak{p}}(x_1,x_2) = \varphi^1_{\mathfrak{p}}(\epsilon^{2-\mathfrak{c}} a(\epsilon^{\mathfrak{c}}) x_1 a(\epsilon^{-\mathfrak{c}}))\varphi^1_{\mathfrak{p}}(x_2).$$

If $\delta > 0$, start with

$$\varphi^0_{\mathfrak{p}}(y) = \begin{cases} \chi(\varpi^{1-\delta} b_y), & \text{if } y \in \begin{bmatrix} \mathfrak{O}_{\mathfrak{P}} & \varpi^{-1-\delta}\mathfrak{o}_{\mathfrak{p}}^{\times} \\ \varpi^{1-\delta}\mathfrak{o}_{\mathfrak{p}} & \mathfrak{O}_{\mathfrak{P}} \end{bmatrix}, \\ 0, & \text{otherwise.} \end{cases}$$

Then the same properties holds.

(*At* $\mathfrak{p} = \mathfrak{P}^2$, $\mathfrak{f} > 0$ *case*): With the same $\varphi^0_{\mathfrak{p}}$ as above, we define, for $\mu = \mu_{\mathfrak{p}}$,

$$\varphi^{\mu}_{\mathfrak{p}}(y) = \mu(\varpi^{1-\delta} y_{(3)})\varphi^0_{\mathfrak{p}}(y); \ \varphi_{\mathfrak{p}}(x_1,x_2) = \varphi^{\mu}_{\mathfrak{p}}(\epsilon^{2-\mathfrak{c}} a^{\mathfrak{c}}_{\epsilon} x_1 a_{\epsilon}^{-\mathfrak{c}})\varphi^1_{\mathfrak{p}}(x_2).$$

This definition is corresponding to the choice of $f \in \sigma$ in (??). For $(t,u) \in \mathbb{U}_{\mathfrak{c}} = \mathfrak{o}_{\mathfrak{p}}^{\times} \times \Gamma_0(\mathfrak{P}^{\mathfrak{c}})$ such as $u = a_t \pmod{\mathfrak{P}^{\mathfrak{f}}}$?needed?,

$$\varphi_{\mathfrak{p}}(\rho(t,u)(x_1,x_2)) = \mu_{\mathfrak{p}}^{-1}(t)\varphi_{\mathfrak{p}}(x_1,x_2), \ \varphi_{\mathfrak{p}}(\rho(t,u')(x_1,x_2)) = \mu_{\mathfrak{p}}(t)\varphi_{\mathfrak{p}}(x_1,x_2). \tag{2.24}$$

(*At* $\mathfrak{p} = \mathfrak{P}$ *inert in* L, $\mathfrak{f} = 0$ *case*): Set $\varphi^0_{\mathfrak{p}}(x) = ch_{(V_{\mathfrak{p}} \cap M_2(\mathfrak{O}_{\mathfrak{p}}))}(a(\varpi^{\delta}) x a(\varpi^{-\delta})) \in \mathcal{S}(V_{\mathfrak{p}})$. Define

$$\varphi_{\mathfrak{p}}(x_1,x_2) = \varphi^0_{\mathfrak{p}}(\varpi^{-\mathfrak{c}} a(\varpi^{\mathfrak{c}}) x_1 a(\varpi^{-\mathfrak{c}}))\varphi^0_{\mathfrak{p}}(x_2)$$

with $\mathfrak{c} = c(\sigma_{\mathfrak{P}})$.

(*At* $\mathfrak{p} = \mathfrak{P}$, $\mathfrak{f} > 0$ *case*): Let $\mu^0 = \mu \cdot ch(\mathfrak{o}_{\mathfrak{p}}^{\times})$. We define

$$\varphi^{\mu}_{\mathfrak{p}}(y) = \mu^0(y_3)\varphi^0_{\mathfrak{p}}(y); \ \ \varphi_{\mathfrak{p}}(x_1,x_2) = \varphi^{\mu}_{\mathfrak{p}}(\varpi^{-\mathfrak{c}} a^{\mathfrak{c}}_{\varpi} x_1 a_{\varpi}^{-\mathfrak{c}})\varphi^1_{\mathfrak{p}}(x_2).$$

For $(t,u) \in \mathbb{U}_{\mathfrak{c}} = \mathfrak{o}_{\mathfrak{p}}^{\times} \times \Gamma_0(\mathfrak{P}^{\mathfrak{c}})$ such as $u = a_t \pmod{\mathfrak{P}^{\mathfrak{f}}}$,

$$\varphi_{\mathfrak{p}}(\rho(t,u)(x_1,x_2)) = \mu_{\mathfrak{p}}^{-1}(t)\varphi_{\mathfrak{p}}(x_1,x_2), \ \varphi_{\mathfrak{p}}(\rho(t,u')(x_1,x_2)) = \mu_{\mathfrak{p}}(t)\varphi_{\mathfrak{p}}(x_1,x_2), \tag{2.25}$$

(*At* $\mathfrak{p}$ *decomposed into* $\mathfrak{p}_1\mathfrak{p}_2$ *in* L): Identifying $\sigma_{\mathfrak{p}_1}, \sigma_{\mathfrak{p}_2}$ with the pair $\sigma_{1\mathfrak{p}}, \sigma_{2\mathfrak{p}}$ as in the previous section, we use the same $\varphi_{\mathfrak{p}}$.

theta lift: Let $GSp_2(k)^N$ denote the elements in $GSp_2(k)$ such that $\nu(g) \in N_{L/k}(L^{\times})$. Using the extended Weil representation, we define, for $g \in GSp_2(\mathbb{A})^N$,

$$\theta(g; f_{\mu}) = \int_{\mathbb{A}^{\times} H^1(k)\backslash H^1(\mathbb{A})} \sum_{x_i \in M_2(k)} (r(g,h)\varphi)(x_1,x_2) f_{\mu}(hh')dh \tag{2.26}$$

where $h' = (1,h'_0) \in H(\mathbb{A})$ is chosen so that $\nu(g) = N_{L/k}\det(h'_0)^{-1}$, and we embed $\mathbb{A}^{\times} \ni t \mapsto (t^2,t) \in H^1(\mathbb{A})$. We consider that f_{μ} is a column vector and φ is a row vector. Since $\theta(g; f_{\mu})$ is left $GSp_2(k)^N$ invariant, we can extend $\theta(g; f_{\mu})$ to a function on $GSp_2(k)\backslash GSp_2(\mathbb{A})$ by insisting that it is left $GSp_2(k)$ invariant and zero outside of $GSp_2(k)GSp_2(\mathbb{A})^N$. We denote by $\theta(g; f_{\mu})$ this extended automorphic form on $GSp_2(\mathbb{A})$. In the similar way to Proposition **??**, or **??**, the following propositions can be shown.

PROPOSITION 2.3 *The highest weight of the representation of* $U_2(\mathbb{C})$ *associated to* $\theta(g; f_{\mu})$ *at* ∞_j *is*

$$\begin{cases} (\kappa_j + 1, 1), & \text{if } L_{\infty_j} \simeq \mathbb{C} \text{ and } \mu_{\infty_j}(-1) = (-1)^{\kappa_j}, \\ (\kappa_j + 1, 0), & \text{otherwise.} \end{cases}$$

PROPOSITION 2.4 *Let* $\mathfrak{e}_{\mathfrak{P}} = c(\sigma_{\mathfrak{P}}), \mathfrak{f}_{\mathfrak{p}} = c(\mu_{\mathfrak{p}})$. $\theta(g, f_\mu)$ *is* $\mu_{\mathfrak{p}}^{-1}$*-semistable on*

$$\begin{cases} K_{\mathfrak{p}}(\mathfrak{e}_{\mathfrak{P}} + 2, \mathfrak{f}_{\mathfrak{p}}), & \text{if } \mathfrak{p} = \mathfrak{P}^2, \\ K_{\mathfrak{p}}(\mathfrak{e}_{\mathfrak{P}}, \mathfrak{f}_{\mathfrak{p}}), & \text{if } \mathfrak{p} = \mathfrak{P} \text{ is innert in } L/k, \\ K_{\mathfrak{p}}(\mathfrak{e}_{\mathfrak{P}_1} + \mathfrak{e}_{\mathfrak{P}_2}, \mathfrak{f}_{\mathfrak{p}}), & \text{if } \mathfrak{p} = \mathfrak{P}_1\mathfrak{P}_2 \text{ decomposed in } L/k. \end{cases}$$

However, remark that in the proposition **??**, the irreducible represention associated to F is

$$\begin{cases} \text{non-holomorphic discrete series representatoin}, & \text{if } L_{\infty_j} \simeq \mathbb{C} \text{ and } \mu_{\infty_j}(-1) = (-1)^{\kappa_j+1}, \\ P_1\text{-induced principal series representation}, & \text{otherwise.} \end{cases}$$

Whitakker function: Let us observe the Whittaker function $W_F = W_F^{1,-1}$ of $F = \theta(g; f_\mu)$ associated to $\psi_{1,-1}$. By the same discussion as the previous section, all the elements in $V(k)^2$ which may contribute to W_F are in the $H^1(k)$-orbit of (v_{-1}, a_{-1}). In this section, we write

$$Z(k) = \{t \times h \in H^1(k) \mid \rho(t,h)v_{-1} = v_{-1}, \rho(t,h)a_{-1} - a_{-1} \in kv_{-1}\} = \{(ad^{\mathfrak{a}}, [\begin{smallmatrix} a & b \\ & d \end{smallmatrix}]) \mid d \in ak^{\times}\},$$

$$Z_0(k) = \{t \times h \in Z(k) \mid \rho(t,h)a_{-1} = a_{-1}\} = \{(ad^{\mathfrak{a}}, [\begin{smallmatrix} a & b \\ & d \end{smallmatrix}]) \mid d \in ak^{\times}, b \in a\epsilon k\}.$$

Remark that $(1, n_{\epsilon y}) \in Z(k)$ for $y \in k$. For every $t \in k_{\mathbb{A}}$, $s(t) = (1, n_{t/2})$ in $Z_0(\mathbb{A})\backslash Z(\mathbb{A})$ uniquely satisfies $\rho(s(t))a_{-1} = a_{-1} + tv_{-1}$. We can also decompose $W_F(1) = \prod_v I_v$ with

$$I_v = \int_{Z_0(k_v)\backslash Z(k_v)} r(1,(t,h))\varphi_v(v_{-1}, a_{-1})\mu(t)W_f(h)d\tilde{h}.$$

Here $d\tilde{h}$ is the Haar measure on $H^1(k_v)$ so that $vol(\mathbb{U}_v) = 1$ for the maximal compact subgroup $\mathbb{U}_v$. We are going to calculate I_v only for v which ramifies in L or innert.

(I_{∞_j} at ∞_j where $L_{\infty_j} \simeq \mathbb{C}$): Noting that all the components of $P(s_1 v_{-1}) = (P_i(s_1 v_{-1}))_{1 \le i \le 2\kappa_j+1}$ are zero except for $i = \kappa_j + 1$, we calculate the coefficient of s_1 of I_{∞_j} as

$$\exp(-2\pi(1-\varepsilon_j^2)) \int_{\mathbb{R}_{>0}} \left(\int_{\mathbb{R}} K_0(4\pi y) \exp(-\pi y^{-2}) \exp(4\pi i x - 2\pi x^2 y^{-2}) dx \right) d^{\times} y$$
$$= \frac{\exp(-2\pi(1-\varepsilon_j^2))}{\sqrt{2}} \int_{\mathbb{R}_{>0}} y K_0(4\pi y) \exp(-\pi(y^{-2} - 2\pi y^2)) d^{\times} y,$$

which is founded to be nonzero since the function integrated is positive for $y > 0$ by (**??**) with $\beta = 0$.

(calculation at $\mathfrak{p} = \mathfrak{P}^2$): For the simplicity we assume the discriminant δ of k over $\mathbb{Q}$ is 0. Let $\mathfrak{e} = ord_{\mathfrak{P}}(c(\sigma))$. Similar to the previous section, we can find at once that every $(t, h) \in H(k_{\mathfrak{p}})$ which may contribute to $I_{\mathfrak{p}}$ is contained in the following double cosets:

i) $Z_0(k_{\mathfrak{p}})(\varpi^m, n(x)a(\varpi^m)^t n(l))\mathbb{U}_{\mathfrak{e}}$ with $x \in k_{\mathfrak{p}}$ and $m \ge 0$ with $l \in \mathfrak{O}/\mathfrak{P}^{\mathfrak{e}}$,

ii) if $\mathfrak{e}$ is even, $Z_0(k_{\mathfrak{p}})(\varpi^{m-\frac{\mathfrak{e}}{2}}, n(x)a(\varpi^m)w(\epsilon^{e}))\mathbb{U}_{\mathfrak{e}}$ with $x \in k_{\mathfrak{p}}$ and $m \ge 0$,
if $\mathfrak{e}$ is odd, $Z_0(k_{\mathfrak{p}})(\varpi^{m+\frac{\mathfrak{e}-1}{2}}, n(x)a(\varpi^m\epsilon)w(\epsilon^{\mathfrak{e}}))\mathbb{U}_{\mathfrak{e}}$ with $x \in k_{\mathfrak{p}}$ and $m \ge 0$.

In the case of even $\mathfrak{e}$, for the i)-type, we suppose that

$$\rho(t,h)(v_{-1}, a_{-1}) = ([\begin{smallmatrix} l\varpi^{-m} & \varpi^{-m} \\ -\varpi l l^{\mathfrak{a}} & -\varpi^{-m} l^{\mathfrak{a}} \end{smallmatrix}], [\begin{smallmatrix} 1 - 2\varpi^{-m} l x & 2\varpi^{-m} x \\ 2l - 2\varpi^{-m} l l^{\mathfrak{a}} x & 1 - 2\varpi^{-m} l^{\mathfrak{a}} x \end{smallmatrix}]) \in supp(\varphi_{\mathfrak{p}})$$

where α is the nontrivial Galois conjugate map of L/k. For this, $m = 0, x \in \mathfrak{o}_{\mathfrak{p}}^{\times}$ and $l \in \mathfrak{P}^{e}$(i.e., $l = 0$) are needed. Among i)-type, the practical contribution is $\chi(-2)G(\chi, \psi)W_f(1)$ by $(1, n_x)$ with $x \in \mathfrak{o}_{\mathfrak{p}}^{\times}$.
For the ii)-type, suppose that

$$\rho(h)(v^{-1}, a^{-1}) = \left(\begin{bmatrix} 0 & 0 \\ \varpi^{e-m} & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2\varpi^{e-m}x & 1 \end{bmatrix}\right) \in supp(\varphi_{\mathfrak{p}}),$$

which is impossible. Summing up, the contribution to $I_{\mathfrak{p}}$ is $\chi(-2)G(\chi, \psi)W_f(1) \neq 0$. The calculations for the odd $\mathfrak{e}$ case and that for inert $\mathfrak{p}$ are similar to this case, so, we omit them.
Novodvorsky zeta integral:

$$Z(s, W, \mu) = \int_{k_{\mathfrak{p}}^{\times}} W(diag[y, y, 1, 1])|y|^{s-\frac{3}{2}}\mu(y)d^{\times}y.$$

This zeta integral is calculated by the same way at the calculation of $W_{F,\mathfrak{p}}(1)$ by replacing $\varphi_{\mathfrak{p}}(x_1, x_2)$ to $\varphi_{\mathfrak{p}}(yx_1, yx_2)$, and supposing that $\det(t^{-1}h_0) = y^{-1}$ for $h = (t, h_0) \in H(\mathbb{A})$. At $\mathfrak{p} = \mathfrak{P}^2$, we calculate

$$W_{F\mathfrak{p}}(diag[y, y, 1, 1]) = \chi(-2)G(\chi, \psi)|y|W_{\mathfrak{P}}(a_{\sqrt{y}})\mu^{-1}(y)$$

where $\sqrt{y}$ is an element of $L_{\mathfrak{P}}^{\times}$ such that $N_{L/k}(\sqrt{y}) = -y$. Remark that $W_{\mathfrak{P}}(a(\sqrt{y})) = W_{\mathfrak{P}}(a(-\sqrt{y}))$ since $\eta_{\mathfrak{P}}(-1) = \mu_{\mathfrak{p}} \circ N_{L/k}(-1) = 1$. Hence $(\chi(-2)G(\chi, \psi))^{-1}Z(s, W_{F\mathfrak{p}})$ is calculated as

$$\int_{k_{\mathfrak{p}}^{\times}} W_{\mathfrak{P}}(a(\sqrt{y}))\mu^{-1}(y)|y|^{s-\frac{1}{2}}d^{\times}y = \int_{L_{\mathfrak{P}}^{\times}} W_{\mathfrak{P}}(a_z)\eta^{-1}(z)|z|^{s-\frac{1}{2}}d^{\times}z = L(s, \tilde{\sigma}_{\mathfrak{P}}).$$

At innert $\mathfrak{p} = \mathfrak{P}$, if there are $z \in L_{\mathfrak{P}}^{\times}$ such as $y = N_{L/k}(z)$,

$$W_{F\mathfrak{p}}(diag[y, y, 1, 1]) = |y|W_{\mathfrak{P}}(a_z)\mu^{-1}(y)$$

and zero otherwise. Hence $Z(s, W_{F\mathfrak{p}})$ is calculated as

$$\int_{N_{L/k}L_{\mathfrak{P}}^{\times}} W_{\mathfrak{P}}(a_z)\mu^{-1}(y)|y|^{s-\frac{1}{2}}d^{\times}y = \int_{L_{\mathfrak{P}}^{\times}} W_{\mathfrak{P}}(a_z)\eta^{-1}(z)|z|^{s-\frac{1}{2}}d^{\times}z = L(s, \tilde{\sigma}_{\mathfrak{P}}).$$

γ**-factor**:

$$Z_N(1-s, \pi(\begin{bmatrix} & -w_1 \\ w_1 & \end{bmatrix})W, \eta_{\mathfrak{p}}\mu^{-1}) = \gamma(s, \pi, \mu, \psi_{1,-1})Z_N(s, W, \mu) \tag{2.27}$$

holds for every $W \in W(\pi, \psi_{1,-1})$. At $\mathfrak{p} = \mathfrak{P}^2$, we calculate

$$\pi(\begin{bmatrix} & -w_1 \\ w_1 & \end{bmatrix})W_{F\mathfrak{p}}(diag[y, y, 1, 1]) = \chi(-2)G(\chi, \psi)|y|\mu^{-1}(y)W_{\mathfrak{P}}(a_{\sqrt{y}}w_1).$$

Hence the left hand side of (**??**) is calculated as

$$\chi(-2)G(\chi, \psi)\int_{k_{\mathfrak{p}}^{\times}} \mu_{\mathfrak{p}}(y)\mu_{\mathfrak{p}}^{-1}(y)W_{\mathfrak{P}}(a_{\sqrt{y}}w_1)|y|^{1+(1-s)-\frac{3}{2}}d^{\times}y = \chi(-2)G(\chi, \psi)\int_{k_{\mathfrak{p}}^{\times}} W_{\mathfrak{P}}(a_{\sqrt{y}}w_1)|y|^{(1-s)-\frac{1}{2}}d^{\times}y$$
$$= \chi(-2)G(\chi, \psi)\int_{L_{\mathfrak{P}}^{\times}} \pi(w_1)W_{\mathfrak{P}}(a_z)|z|^{(1-s)-\frac{1}{2}}d^{\times}z = \chi(-2)G(\chi, \psi)Z(1-s, \pi(w^{-1})W_{\mathfrak{P}}).$$

At innert $\mathfrak{p} = \mathfrak{P}$, if there are $z \in L_{\mathfrak{P}}^{\times}$ such as $y = N_{L/k}(z)$,

$$\pi(\begin{bmatrix} & w_1 \\ w_1 & \end{bmatrix})W_{F\mathfrak{p}}(diag[y, y, 1, 1]) = |y|\mu^{-1}(y)W_{\mathfrak{P}}(a_zw_1)$$

and zero otherwise. Hence the left hand side of (??) is calculated as

$$\int_{N_{L/k}L_{\mathfrak{P}}^{\times}} \mu_{\mathfrak{p}}(y)\mu_{\mathfrak{p}}^{-1}(y)W_{\mathfrak{P}}(a_z w_1)|y|^{1+(1-s)-\frac{3}{2}}d^{\times}y = \int_{L_{\mathfrak{P}}^{\times}} W_{\mathfrak{P}}(a_z w_1)|z|^{(1-s)-\frac{1}{2}}d^{\times}z$$
$$= \int_{L_{\mathfrak{P}}^{\times}} \pi(w_1)W_{\mathfrak{P}}(a_z)|z|^{(1-s)-\frac{1}{2}}d^{\times}z = Z(1-s,\pi(w_{-1})W_{\mathfrak{P}}).$$

Thus, with noting $W_{\mathfrak{P}} \in \tilde{\tilde{\sigma}} = \sigma$, $\gamma(s,\pi,\psi_{1,-1}) = \gamma(s,\tilde{\sigma},\psi_{L})$ at every finite $\mathfrak{P}$.

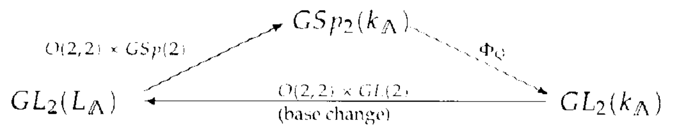
L-**factor,** ε-**factor**: By the above results, we can also conclude $L(s,\pi_{\mathfrak{p}},\xi) = L(s,\tilde{\sigma}_{\mathfrak{P}},\xi \circ N_{L/k})$, $\varepsilon(s,\pi,\xi,\psi_{1,-1}) = \varepsilon(s,\tilde{\sigma}_{\mathfrak{P}},\xi \circ N_{L/k})$ for every $\xi \in \widehat{k}_{\mathfrak{p}}$, by the same way as in the previous section.

cuspidality: In case that σ is obtained by the base change lift from $\sigma_0 \in \Pi(GL_2(k_{\mathbb{A}}))$ and $\mu = \chi\eta_{\sigma_0}$, F is non-cuspidal since $L_S^{st}(s,F)$ has a double pole at $s = 1$. Indeed, the image $\Phi_Q(F;g_1)$ by the Siegel operator Φ_Q defined section **??** at some $g_1 \in Sp_2(\mathbb{A})$ is not zero. Indeed, we can regard $\Phi_Q(F;g_1)$ as a theta lift from $GO(2,2)$ to $GL(2)$ similar to the $\sigma_1 = \sigma_2$ case in the previous section. This phenomnenon implies that

$$\int_{GL_2(k)\backslash GL_2(k_{\mathbb{A}})} \mu^{-1}(\det(h))f_{\sigma}(h)d^{\times}h \neq 0.$$

Thus f_{μ} can be regarded as a member of an *"distingwised' representation* in an extended sence (c.f. Flicker [?]). Moreover this result will give an answer for the problem when $\pi \in \Pi GL_2(L_{\mathbb{A}})$ is distingwished, that is, if and only if π is obtained by the base change lift, π is distngwished (or in the extended sence). In a nearly future, we will return to this topic.

However, different from the $\sigma_1 = \sigma_2$ case in the previous section, this is a defective rotation:



In general, when one makes a round from $GL_2(k_{\mathbb{A}})$, the result may not be an eigenform at $\mathfrak{p}$ where $\mathfrak{p} = \mathfrak{P}$ is innert in L/k (see the next section for the explanation).

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