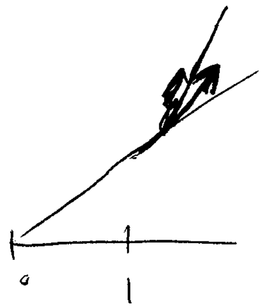

Assume that $g(s)$ satisfies the following convexity property:

$$\frac{g(s)-g(0)}{s} \leq g'(s), \text{ i.e. } g(0) \geq g(s) - sg'(s),$$

for $s \geq 1$. Of course, there is no reasonable reason to believe this since $g(s)$ is maybe not even defined for all $0 < s < 1$. However, this holds true, as I shall show in a moment. Taking exponentials we obtain from this

$$\frac{R}{w} \geq \frac{s(s-1)}{2^{r_1}} D(s) \exp\Big(-1 - \frac{s}{s-1} - s\frac{\gamma'}{\gamma}(s) - s\frac{D'}{D}\Big)$$

Using

$$D(s) > 1, \quad -\frac{D'}{D}(s) > 0$$

we obtain an absolute lower bound for the $\mathrm{Reg}(L)$:

$$\frac{R}{w} \geq \frac{s(s-1)}{2^{r_1}\mathrm{e}} \gamma(s) \exp\Big(-\frac{s}{s-1} - s\frac{\gamma'}{\gamma}(s)\Big).$$

Choosing $s = 4/3$ we finally find

Theorem. *(Skoruppa 93)*

$$\mathrm{Reg}(L) \geq 0.00299 \cdot \exp(0.48r_1 + 0.06r_2).$$

It remains to prove the convexity property.

Lemma. *Assume that $f : \mathbb{R}_+ \to \mathbb{R}$ satisfies $f \geq 0$, $Z(s) := \int_0^\infty f(t)t^s\frac{dt}{t} < \infty$ for $s > 1$ and that $t(C + f(t))$ is increasing for some constant C. Then*

$$C \geq \frac{s(s-1)}{e} Z(s) \exp\Big(-\frac{s}{s-1} - s\frac{Z'}{Z}(s)\Big).$$

Proof. Set

$$\log_+(t) := \log\max(1,t), \quad w(t) := t\log_+(1/t).$$

The function

$$I(a) := \int_0^\infty (C + f(t))t\,w\big((at)^{s-1}\big)\,\frac{dt}{t}$$

is decreasing in a (set $t/a \mapsto t$). Note

$$w'(t) = \begin{cases} -(1+\log t) & t < 1 \\ 0 & \textit{otherwise} \end{cases}.$$