Regulators

The most mysterious and subtle objects in number fields are possibly the units. They show a rather paradox behaviour. Sometimes the have a rather high complexity:

$$K = \mathbb{Q}(\sqrt{611953}), \quad \mathbb{Z}_K^* = \pm\varepsilon^{\mathbb{Z}}, \quad x^2 - 3 \ldots 499 \text{ digits} \ldots 8x - 1.$$

Sometimes the have a very small complexity, which makes them also difficult to handle: For the height

$$H(\alpha) = \prod_{\alpha' \text{conjugate to } \alpha} \max(1, |\alpha|)$$

of an algebraic number (here integer) it is conjectured (Lehmer) that there is an absolute constant $C > 1$ such that

$$H(\alpha) \geq C$$

for all α which are not roots of unity. The numbers with very small height are the non-reciprocal units. An important measure for the group of units of a number field are the regulators. We consider here even relative regulators. Let L/K be an extension of number fields. We let

$$E = E_{L/K} := \{\varepsilon \in \mathbb{Z}_L : N_{L/K}(\varepsilon) = \text{ root of unity}\},$$

the group of relative units. It is easy to show that the rank of E is

$P_K \ni v \rightsquigarrow \tilde{v} \in P_L$ pick

$$\operatorname{rank}(E) = \#P_L - \#P_K,$$

$P_L \setminus \{\tilde{v} \mid v \in P_K\}$: $v_1, \ldots, v_s$

where P_K is the set of places of K (and similarly for P_L). For each place v of K pick a place $\widetilde{v}$ of L extending v, and let v_j $(1 \leq j \leq s)$ be the places of L different from the $\widetilde{v}$. Set $e_v = 1$ or $e_v = 2$ accordingly as v is real or complex. The relative regulator is the defined to be

$$\operatorname{Reg}(L/K) := \left|\det\left(\log|\varepsilon_h|_{v_k}^{e_{v_k}}\right)_{1\leq h,k\leq s}\right|$$

where the absolute values $|\alpha|_v$ associated to v are normalized such that

they are [illegible] of $|\cdot| \sim p$.

$$|N_{L/Q}(\alpha)| = \prod_{v \in P_L} |\alpha|_v^{e_v}.$$

If ε is a Salem number, i.e. $\varepsilon > 0$, $1/\varepsilon$ is conjugate to ε and all other conjugates $\varepsilon_1, \ldots, \varepsilon_p$ have absolute value 1, i.e. an interesting unit in the context of the Lehmer conjecture, then

$$\operatorname{Reg}(\mathbb{Q}(\varepsilon)/\mathbb{Q}(\varepsilon + 1/\varepsilon)) = \frac{n}{2}\log|\varepsilon|$$