Let L be a lattice in euclidean n-space. Set

$$\theta(t) = \theta_L(t) = \mathrm{e}^{-\pi t x^2} = 1 + a(L)t^{-\pi r t} + \dots,$$

where r is the smallest non-zero length of points in L. Now,

$$\theta(t) \quad \downarrow$$

is obviously *de*creasing in t. However,

$$t^{n/2}\theta(t) \quad \uparrow$$

is *in*creasing. Indeed, this follows on using Poisson summation

$$\theta(t)t^{n/2} = \det(L)^{-1}\theta_{L*}(1/t)$$

On taking the derivative we find

$$0 \leq \frac{d}{dt}\theta(t)t^{n/2} = \sum_{x\in L}\big(-\pi t x^2 + \frac{n}{2}\big)t^{n/2-1}\mathrm{e}^{-\pi t x^2}.$$

Assume that the smallest nonsero length in L is 1. Then, for

$$u := \pi t \geq \frac{n}{2},$$

all terms in the last sum but the first are negative, and on dropping those with $x^2 > 1$ we obtain

$$a(L)(u - \frac{n}{2})\,\mathrm{e}^{-u} \leq \frac{n}{2}.$$

The left hand side takes its maximum for $u = \frac{n}{2} + 1$ (which is in the valid range), and hence

$$a(L) \leq \frac{n}{2}\mathrm{e}^{\frac{n}{2}+1}.$$

For example $a(L) \leq 2.2, 7.3, 18.2$ for $n = 1, 2, 3$ which is not too bad. In any case we have already proved

$$\lambda_n \leq \mathrm{e}^{\frac{n}{2}(1+o(1))} \leq 1.65^{n(1+o(1))}.$$

By the way, the same proof would also yield the following amusing (though trivial) result:

An (elliptic) cusp form has positive as well as negative Fourier coefficients.

I shall indicate in a moment how a refinement of the above method could be used to prove a much sharper asymptotic bound. The idea is roughly to replace the exponential function in the definition of θ by another function which reduces the error made when discarding the terms with $x^2 > 1$. However, for this it is then sometimes easier to pass to Dirichlet series, i.e. to apply Mellin transformation. This is for the simple reason that the higher transcendental functions have usually simple Mellin transforms: ratios of Gamma functions with shifted elements. We illustrate these techniques in the context of another problem.