• Starting point (second topic of lecture):

$$\vartheta_L(t) = \sum_{x\in L} e^{-\pi t x^2} = \sum_{i=0}^{\infty} a_i(L)\, e^{-\pi t \lambda_i}$$

($0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots$ the square-lengths of vectors in L, $a_i(L) = \#\{x\in L \mid x^2 = \lambda_i\}$.)

Fact: $\vartheta_L(t)\, t^{n/2}$ mon. increasing (note $\vartheta_L(t) \downarrow$)

(Poisson summation: $\vartheta_L(t)\, t^{n/2} = (\det L)^{-1} \vartheta_{L^*}(\frac{1}{t})$. $\vartheta_{L^*}(t)\downarrow$, hence $\vartheta_{L^*}(\frac{1}{t})\uparrow$.)

$$0 \leq \frac{d}{dt} \vartheta_L(t)\, t^{n/2} = \sum_{n=0}^{\infty} e^{-\pi t \lambda_n} a_n(L) \left(t^{n/2-1}\frac{n}{2} - \pi \lambda_n t^{n/2}\right)$$

$$\cancel{t^{\frac{n}{2}-1}\frac{n}{2}}$$

Dropping all terms $n \geq 1$ for $\frac{n}{2} - \pi t \lambda_1 \leq 0$ gives

$$1 \geq \left(\frac{\pi \lambda_1 t}{n/2} - 1\right) a_1(L)\, e^{-\pi t \lambda_1}$$

max in t for $t = \frac{\frac{n}{2}+1}{\pi \lambda_1}$:

$$a_1(L) \leq \frac{n}{2} e^{\frac{n}{2}+1}$$

in particular

$$\lambda_n \leq e^{\frac{n}{2}(1+o(1))} = 1.64^{n(1+o(1))}.$$

Improvements — take Mellin transforms:

$$Z(s) = \pi^{-\frac{ns}{2}} \Gamma(\tfrac{ns}{2})\, D_L(s), \quad D_L(s) = \sum_{\substack{x\neq 0\\ x\in L}} \frac{1}{(x^2)^{\frac{ns}{2}}} = a_1(L)\,\lambda_1^{-s} \dots$$

analytic in $\mathbb{C}$ except poles at $0, 1$:

$$\operatorname{Res}_{s=0} Z(s) = -1/\tfrac{n}{2}$$