To identify ρ (or $\eta^k \otimes \rho$):

$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\rho(T)$ = diag. matrix w. roots of 1 on diag., $\rho(T)^N = id$ some N

Ker ρ congr. subgroup

<u>Theorem</u> [Fricke-Klein-Wohlfart]

$\rho : \Gamma \to GL(n,\mathbb{C})$ any rep., Ker ρ = congruence subgroup.

Let $N \in \mathbb{Z}_{>0}$ s.t. $\rho(T)^N = id$.

Then Ker $\rho \supseteq \Gamma(N)$.

Thus ρ factors to $\underline{\rho} : \Gamma/\Gamma(N) = SL(2, \mathbb{Z}/N\mathbb{Z}) \longrightarrow GL(n,\mathbb{C})$.

Only finitely poss. for ρ. To further reduce possibilities:

$\eta^k \xi$ has rat. Fourier coeff.

<u>Theorem</u>

$\rho : \Gamma \to GL(n,\mathbb{C})$ rep., Ker $\rho \supseteq \Gamma(N)$ some $N \in \mathbb{Z}_{>0}$, let $K = \mathbb{Q}(e^{2\pi i/N})$.

~~If there~~ Assume there ex. $F \in M_k(\rho)$ (some k) such that F has Four. coeff. in K.

Then $\rho(\Gamma) \subseteq GL(n,K)$.

Using this ρ was reduced in all cases to one ~~or~~ two poss.
To see how much freedom is left one has to look at the dimension of the spaces $M_k(\rho)$.