Theorem [Eholzer - ~]

let c as in the Tables, H_c set of corresp. conf. dim.

Assume there exist ξ_h $(h \in H_c)$ satisfying (1) – (5)

and such that ξ_h is inv. under some congruence subgroup

of Γ. ~~$\xi_h(A\tau) = \xi$~~

Then the functions ξ_h are unique (up to multiplic. by scalar)

Three main ingredients in proof

(i) first

Let ξ = column vector of the ξ_h (some fixed order), then

$$\xi(A\tau) = \varrho(A)\,\xi(\tau) \quad \text{for a rep. } \varrho : \Gamma \longrightarrow GL(n, \mathbb{C}) \quad (n = \# H_c)$$

vector valued modular function on Γ

useful to multiply by powers of

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$$

then we land in a

space of vector valued modular forms on Γ (poss. of half-integral wt)

$$\tilde{\Gamma} = \{ (A, w(\tau)) \mid A \in \Gamma,\ \cancel{w = \ldots}\ w \in \mathrm{Hol}(\mathfrak{H}),\ w^2 = c\tau + d \}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

group: $(A, w) \cdot (A', w') = (AA', w(A'\tau)\, w'(\tau))$

action: $F : \mathfrak{H} \longrightarrow \mathbb{C}^n$, $(A, w) \in \tilde{\Gamma}$: $\quad F|_k (A, w) = F(A\tau)\, w(\tau)^{-2k}$, $\quad k \in \frac{1}{2}\mathbb{Z}$

$$\sigma : \tilde{\Gamma} \longrightarrow GL(n, \mathbb{C}) \quad \text{representation}$$

$$M_k(\sigma) = \left\{ F : \mathfrak{H} \xrightarrow{\text{hol.}} \mathbb{C}^n \;\middle|\; \begin{array}{l} F|_k \alpha = \varrho(\alpha) F \quad \forall \alpha \in \tilde{\Gamma} \\ F = O(1) \text{ for } \operatorname{Im}\tau \to \infty \end{array} \right\}$$

basic example:

$$\eta \in M_{\frac{1}{2}}(\theta) \quad \text{for a } \theta : \tilde{\Gamma} \longrightarrow \mu_{24}$$

$$\eta^k \xi \in M_{\frac{k}{2}}(\theta^k \otimes \varrho) \quad \text{for } \cancel{k \text{ big enough}} \quad k \geq \tilde{c}$$