$$\frac{Q\circ A + Q\circ y^2 A_y}{y^2} = 2(-ay^2+c), \qquad \widehat{Q\circ y^2 A_y(iy)} = \widehat{Q\circ A(iy)} = ay^2+c,$$

thus

$$C(\Delta, v)$$

$$= \int_0^\infty 2i\sqrt{v} \sum_{\substack{Q\in \mathcal{L}(\Delta\Delta_0, \nu r_0)\circ A \\ Q=(a,b,c)}} \chi_{\Delta_0}(Q\circ A^{-1}) \left(-ay + \frac{c}{y}\right) e^{-\lambda\left(ay+\frac{c}{y}\right)^2} \frac{dy}{y}.$$

$$= 2i\sqrt{v}\Big\{ \sum_{\substack{Q \\ ac\neq 0}} \chi_{\Delta_0}(Q\circ A^{-1}) \underbrace{\int_0^\infty \left(-ay+\frac{c}{y}\right) e^{-\lambda\left(ay+\frac{c}{y}\right)^2}\frac{dy}{y}}_{I}.$$

$$+ \sum_{\substack{a \bmod m|\Delta_0|,\ b\in\mathbb{Z} \\ (a,b,0)\in \mathcal{L}\circ A}} \chi_{\Delta_0}(Q\circ A^{-1}) \underbrace{\int_0^\infty \sum_{\substack{x\in\mathbb{Z} \\ x\equiv a\ (m|\Delta_0|)}} (+x)\, e^{-\lambda x^2 y^2}\, dy}_{J}$$

$$+ \sum_{\substack{c \bmod m|\Delta_0|,\ b\in\mathbb{Z} \\ (0,b,c)\in\mathcal{L}\circ A}} \chi_{\Delta_0}(Q\circ A^{-1}) \underbrace{\int_0^\infty \sum x\, e^{-\lambda x^2/y^2}\, \frac{dy}{y^2}}_{K}.$$

To evaluate I set

$$y = \sqrt{\left|\tfrac{c}{a}\right|}\, e^{\theta}, \quad \text{then}$$

$$I = -\operatorname{sign}(a)\sqrt{|ac|} \int_{-\infty}^{+\infty} e^{-\lambda|ac|\, c(\theta)^2}\, d\,c(\theta)$$

where

$$c(\theta) = \begin{cases} 2\cosh\theta & ac>0 \\ 2\sinh(\theta) & ac<0 \end{cases}.$$

Then

$$I = \begin{cases} 0 & ac>0 \\ -\operatorname{sign}(a)\sqrt{|ac|}\sqrt{\dfrac{\pi}{\lambda|ac|}} = -\operatorname{sign}(a)\dfrac{\sqrt{\pi|\Delta|}}{\sqrt{v}} \end{cases}$$