Now instead of dealing with the abstract Hom(,) it is better to deal with vectors, i.e. with a concrete vector space. For this we rewrite the above Lemma using the so-called Manin trick:

<u>Lemma</u> Let $\varepsilon \in \{\pm 1\}$. Then

$$f \longmapsto \{r_A^\varepsilon(f)\}_{A \in \Gamma_0(m)\backslash SL_2(\mathbb{Z})}$$

$$r_A^\varepsilon(f) := \int_0^{i\infty} f(A\tau)\, dA\tau - \varepsilon \int_0^{i\infty} f(gAg\tau)\, d(gAg)\tau$$

$\left(g = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} \right.$, then $\left. g\begin{pmatrix} a & b \\ c & d \end{pmatrix} g = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \right)$

defines an injection

$$S_2(\Gamma_0(m)) \hookrightarrow \mathbb{C}^{\Gamma_0(m)\backslash SL_2(\mathbb{Z})}$$

<u>Proof</u> Express $\pi_f^\varepsilon(B)$ $(B \in \Gamma_0(m))$ in terms of the $r_A^\varepsilon(f)$.

Write $B_{\cancel{r}} = \pm T^{n_1} S T^{n_2} S \dots T^{n_r} S$ $\quad \left(T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix};\right.$ recall $\left. SL_2(\mathbb{Z}) = \langle S, T \rangle \right)$

Set

$$B_j = \pm T^{n_1} \dots T^{n_j} S, \cancel{B_0 = 1}.$$

Then

$$\int_0^{B0} = \int_0^{B_1 0} + \int_{B_1 0}^{B_2 0} + \dots + \int_{B_{r-1} 0}^{B_r 0},$$

then

$$\int_0^{B0} f(\tau)\,d\tau = \int_0^{B_1 0} f(\tau)\,d\tau + \int_0^{B_1^{-1} B_2 0} f(B_1 \tau)\, d(B_1 \tau) + \dots + \int_0^{B_{r-1}^{-1} B_r 0} f(B_{r-1}\tau)\, d B_{r-1}\tau$$

Note that $B_{j-1}^{-1} B_j 0 = T^* S 0 = T i\infty = i\infty$.

Do same with gBg, obtain

$$\pi_f^\varepsilon(B) = r_1^\varepsilon(f) + r_{B_1}^\varepsilon(f) + \dots + r_{B_{r-1}}^\varepsilon(f). \quad \square$$