4. How to crack Jac form II

Intermezzo on periods of modular forms:

Lemma The map
$$f \longmapsto \pi_f\,, \quad \pi_f(A) := \int_0^{A0} f(\tau)\,d\tau \quad (A \in \Gamma_0(m))$$
defines an injection
$$S_2(\Gamma_0(m)) \hookrightarrow \mathrm{Hom}(\Gamma_0(m), \mathbb{C}) \quad \left(=: H^1(\Gamma_0(m), \mathbb{C})\right)$$

Proof
Set $F(t) := \int_0^t f(\tau)\,d\tau$. Then for $A \in \Gamma_0(m)$ you have
$$F(At) = \int_0^{At} f(\tau)\,d\tau = \pi_f(A) + \int_{A0}^{At} f(\tau)\,d\tau = \pi_f(A) + \int_0^t f(A\tau)\,dA\tau$$
$$= \pi_f(A) + F(t).$$
From this it follows that (1) $\pi_f : \Gamma_0(m) \to \mathbb{C}$ is a homomorphism.
(2) $\pi_f(A) = 0 \;\forall A \in \Gamma_0(m) \Rightarrow F \in S_0(\Gamma_0(m))$, i.e. $F = 0$, i.e. $f = 0$,
i.e. π_f is injective.

Actually one has
$$(*) \qquad \overline{S_2(\Gamma_0(m))} \oplus S_2(\Gamma_0(m)) \xrightarrow{\cong} \mathrm{Hom}(\Gamma_0(m), \mathbb{C})_{\mathrm{cusp}}$$
But instead of considering antiholomorphic mod. forms it is nicer to ~~consider~~ transport the splitting from the left side of (*) to the right side and to formulate (*) as two isomorphisms of $S_2(\Gamma_0(m))$ with the corresponding splitting of $\mathrm{Hom}(\ ,\ )$. The result is:

Lemma Let $\varepsilon \in \{\pm 1\}$. Then
$$f \longmapsto \pi_f^{\varepsilon}\,, \quad \pi_f^{\varepsilon}(A) = \left(\int_0^{A0} - \varepsilon \int_0^{-A0}\right) f(\tau)\,d\tau$$
defines injections
$$S_2(\Gamma_0(m)) \hookrightarrow \mathrm{Hom}(\Gamma_0(m), \mathbb{C}).$$