Note that ~~$\Gamma_0(m)$~~ $\mathcal{L}(\Delta, r)$ is invariant under the action $\Gamma_0(m)$ ~~~~ w.r.t.

$$\circledast (Q, A) \to (Q\circ A)(t) := Q\left(\frac{at+b}{ct+d}\right)(ct+d)^2 .$$

Lemma

$$\overline{\vartheta(\tau, z; At)(ct+d)^{-2}} = \overline{\vartheta(\tau, z; t)} \quad \forall A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(m).$$

~~proof~~ Easy exercise using $\circledast$, $\frac{\hat{Q}(At)}{(\operatorname{Im} At)^2} = \widehat{Q\circ A(t)}$, $\chi_{\Delta_0}(Q\circ A) = \chi_{\Delta_0}(Q)$.

Proposition Let $\phi \in S_{2,m}^{\operatorname{sign}(\Delta_0)}$; then

$$f(t) := \langle \phi \mid \vartheta_{\Delta_0, r_0}(\cdot, t)\rangle = \left(\frac{\Delta_0}{0}\right) \operatorname{Cont}(\phi) + \sum_{\ell=1}^{\infty}\left(\sum_{a \mid \ell}\left(\frac{\Delta_0}{a}\right) c_\phi\left(\Delta_0 \frac{\ell^2}{a^2}, r_0 \frac{\ell}{a}\right)\right) e^{2\pi i \ell t}.$$

In particular, $f \in M_2(\Gamma_0(m))$.

~~proof~~ lengthy, tricky computation

Corollary: For each Δ_0, r_0, $\Delta_0 \equiv r_0^2 \bmod 4m$ one has a map

$$\mathcal{S}_{\Delta_0, r_0} : S_{2,m}^{\operatorname{sign}\Delta_0} \longrightarrow M_2(\Gamma_0(m))$$

given by

$$\mathcal{S}_{\Delta_0, r_0}(\phi) = \sum_{\ell=1}^{\infty}\left(\sum_{a \mid \ell}\left(\frac{\Delta_0}{a}\right) c_\phi\left(\Delta_0 \frac{\ell^2}{a^2}, r_0 \frac{\ell}{a}\right)\right) e^{2\pi i \ell t}.$$

Main Theorem: There exists ~~a~~ linear combination of the $\mathcal{S}_{\Delta_0, r_0}$ which define Hecke-equivariant isomorphisms

$$J_{k,m}^{+} \oplus J_{k,m}^{-} \xrightarrow{\cong} \mathfrak{M}_{2k-2}(\Gamma_0(m)),$$

where $\mathfrak{M}_{2k-2}(\Gamma_0(m))$ is a "canonical" subspace of $M_{2k-2}(\Gamma_0(m))$ containing all newforms of level m and an "intelligent" choice of oldforms.

~~proof~~ 60 pages.