3. Lifting of Jacobi forms to modular forms

Fix $\Delta_0, r_0 \in \mathbb{Z}$, $\Delta_0 \equiv r^2 \bmod 4m$, Δ_0 = fundamental discriminant (i.e. there $\nexists$ ~~nontrivial~~ existing suitable n, $n^2 \mid \Delta_0$, $\Delta_0/n^2 \equiv 0,1 \bmod 4$).

Set

$$\vartheta_{\Delta_0,r_0}(\tau,z;t) := \sum_{\substack{\Delta, r \in \mathbb{Z} \\ \Delta \equiv r^2 \bmod 4m}} C_v(\Delta,r;t)\, e^{2\pi i\left(\frac{r^2-\Delta}{4m}u + \frac{r^2+\sigma\Delta}{4m} iv + rz\right)}$$

$$C_v(\Delta,r;t) = v^{1/2} \sum_{Q \in \mathcal{L}(\Delta\Delta_0, rr_0)} \chi_{\Delta_0}(Q)\, \frac{Q(t)}{\eta^2}\, \exp\left(-\frac{\pi v \hat{Q}(t)^2}{m|\Delta_0|\eta^2}\right)$$

where

$\sigma = \mathrm{sign}(\Delta_0)$

$\tau = u+iv$, $t = \xi + i\eta \in \mathfrak{h}$, $z \in \mathbb{C}$,

$\mathcal{L}(\Delta,r) = \{at^2+bt+c \in \mathbb{C}[t] \mid a,b,c \in \mathbb{Z},\ m \mid a,\ b \equiv r\ (2m)\}$

$Q = at^2+bt+c$, then $\hat{Q}(t) = a|t|^2 + b\xi + c$

$$\chi_{\Delta_0} : \bigcup_{\substack{\Delta, r \in \mathbb{Z} \\ \Delta \equiv r^2 \bmod 4m \\ \Delta_0 \mid \Delta,\ \Delta/\Delta_0 \equiv 0,1\,(4)}} \mathcal{L}(\Delta,r) \longrightarrow \{\pm 1\}, \qquad \chi_{\Delta_0}([ma,b,c]) = \begin{cases} \left(\frac{\Delta_0}{n}\right) & \text{if } \Delta_0 \mid b^2-4mac,\ (b^2-4mac)/\Delta_0 \equiv 0,1 \bmod 4,\ (a,b,c,\Delta_0)=1 \\ 0 & \text{otherwise} \end{cases}$$

n = any integer, prime to Δ_0, represented by one of the quad. forms $m_1ax^2+bxy+m_2cy^2$ for some decomp. $m = m_1m_2$, $m_i > 0$. (Theorem: $\exists$ such n and $\left(\frac{\Delta_0}{n}\right)$ independent of the specific choice of n)

Lemma $\vartheta_{\Delta_0,r_0}(\tau,z;t)$ transforms like an element of $J^{\mathrm{sign}(\Delta_0)}_{k,m}$ in (τ,z).

Proof For $\Delta_0 = 1$ immediately from ~~the~~ above theorem.
For $\Delta_0 \neq 1$ from the other theorem (+ properties of χ_{Δ_0} — not that immediately.)