$C_\phi(D,r)$ depends on r only modulo $4m$,

$C_\phi(D,r) \neq 0$ at most for $\begin{cases} D \leq 0 & \text{in the holomorphic case} \\ D \gtrless 0 & \text{in the skew-hol. case} \end{cases}$

(iii) $$\phi\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) e^{-2\pi i m \frac{z^2}{\tau}} = \phi(\tau,z) \cdot \begin{cases} \tau^k & \text{in the holomorphic case} \\ \overline{\tau}^{k-1} |\tau|^{+1} & \text{in the skew-holomorphic case.} \end{cases}$$

$J^-_{k,m}$, $J^+_{k,m}$ = space of holomorphic, skew-holomorphic Jacobi forms.

$\phi(\tau,z)$ is called cusp form if $C_\phi(D,r) = 0$ for $D = 0$.

~~Holomorphic~~ Remarks $S^-_{k,m}, J^+_{k,m}$ = ~~cusp~~ Jacobi cusp forms.

① $\phi \in J^-_{k,m} \Rightarrow \phi$ holomorphic in τ

② Definition of "Jacobi form" can be rewritten purely in terms of a certain invariance under $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 \subseteq J(\mathbb{R}) = SL_2(\mathbb{R}) \ltimes \mathbb{R}^2 \cdot S^1$ (discrete)

Examples

① Quotient of two Jacobi forms of same index for fixed τ is doubly periodic in z with respect to $\mathbb{Z}\tau + \mathbb{Z}$,

e.g. $$\wp(\tau,z) = \frac{1}{z^2} + \sum_{\substack{\gamma \in \mathbb{Z}\tau+\mathbb{Z} \\ \gamma \neq 0}} \frac{1}{(z-\gamma)^2} - \frac{1}{\gamma^2} = \frac{\phi_{12,1}}{\phi_{10,1}}$$

where $S^-_{10(12),1} = \mathbb{C}\cdot\phi_{10(12),1}$.

② Weierstrass σ-function

$$\sigma(\tau,z) = z \prod_{\substack{\gamma \in \mathbb{Z}\tau+\mathbb{Z} \\ \gamma \neq 0}} \left(1 - \frac{z}{\gamma}\right) e^{+\frac{z}{\gamma} + \frac{1}{2}\frac{z^2}{\gamma^2}}$$

transforms like holomorphic Jacobi form of weight -1, index $\frac{1}{2}$ (not quite true)

③ $J^+_{*,1} := \bigoplus_k J^+_{k,1}$ = ~~$M_* \cdot A \oplus M_*$~~ free module of rank 2 over $M_* := \bigoplus_k M_k(SL_2(\mathbb{Z}))$ via $(f(\tau), \phi(\tau,z)) \mapsto f(-\overline{\tau})\phi(\tau,z)$