Now the φ_m belong to the space $J_{k,m}$ of Jacobi forms of index m and weight k, i.e. to the space

$$J_{k,m} = \left\{ \phi : \mathfrak{H}\times\mathbb{C} \xrightarrow{\text{hol.}} \mathbb{C} \;\middle|\; \begin{array}{l} \phi|_{k,m}\gamma = \phi \quad \forall \gamma \in \Gamma_1^J \\ \phi(\tau,z) = \sum\limits_{\substack{n,r\in\mathbb{Z},\, n\geq 0 \\ r^2-4mn\leq 0}} c(n,r)\, e^{2\pi i(n\tau+rz)} \end{array} \right\}.$$

Thus a Jacobi form is a mixture between an elliptic modular form and a theta function associated to the elliptic curve $\mathbb{C}/\mathbb{Z}\tau+\mathbb{Z}$ for fixed τ.

The theory of Jacobi forms originated in 1980 ~~[illegible]~~ by work of Eichler-Zagier and is very well developped meanwhile. The basic notions that we need are:

$J_{k,m}^{cusp}$ = subspace of cusp forms $\left(c(n,r)=0 \text{ unless } r^2-4mn<0\right)$, $F \in S_k(\Gamma_2) \Rightarrow \varphi_m$ cusp form.

$\langle \phi, \psi \rangle$ = Petersson scalar product $= \int_{\Gamma_1^J\backslash\mathfrak{H}\times\mathbb{C}} \phi\,\overline{\psi}\, e^{-4\pi m \frac{y^2}{v}}\, v^k\, dV$ (dV = Γ_1^J invariant volume element of $\mathfrak{H}\times\mathbb{C}$ such that $\int_{\Gamma_1^J\backslash\mathfrak{H}\times\mathbb{C}} dV = 1$)

~~Hecke~~ There exists a Hecke-theory, i.e. Hecke-operators $T(\ell)$ $(\ell, ?, -)$

Main theorem: $J_{k,m} \underset{\text{equivariantly}}{\overset{\text{Hecke}}{\approx}}$ certain subspace of $M_{2k-2}(\Gamma_0(m))$.

Thus, as indicated here, Jacobi forms are very well understood, much better than Siegel modular forms. Hence it would be a good idea to set up the following program:

- Study Siegel forms of degree two via their Fourier Jacobi development.