Except for the listed facts very few is known about the nature of the $\lambda(\ell)$. For example the following question is still unsettled:

— Do the $\{\lambda(\ell)\}_{\ell=1,2,\ldots}$ uniquely determine the eigenform F?

There are at least two subspaces of $M_k(\Gamma_2)$ which are very well understood:

~~There is a canonical Hecke invariant~~ splitting.

$$\cancel{M_k(\Gamma_2) = M_k^{Eis}(\Gamma_2) \oplus M_k^{Maaß}(\Gamma_2)} \oplus M_k^{?}$$

Theorem (Klingen, Maaß, Evdokimov, Andrianov, ... summary of results of ...) Let k be even. Set

$M_k^{Eis}(\Gamma_2)$:= space spanned by the Klingen-Eisenstein series

$S_k^{Maaß}(\Gamma_2)$:= Maaß space $M_k^{Maaß}(\Gamma_2)$ $\cap$ $S_k(\Gamma_2)$

$S_k^{2}(\Gamma_2)$:= space spanned by all eigenforms in $S_k(\Gamma_2)$ such that $Z_F(s)$ is holomorphic

Then these spaces are all Hecke invariant and one has

$$M_k(\Gamma_2) = M_k^{Eis}(\Gamma_2) \oplus S_k^{Maaß}(\Gamma_2) \oplus S_k^{2}(\Gamma_2).$$

There exist Hecke-equivariant isomorphisms such that:

$M_k^{Eis}(\Gamma_2) \cong M_k(SL_2(\mathbb{Z}))$ $F \leftrightarrow f$ then $Z_F(s) = L_f(s)L_f(s-k+2)$

$S_k^{Maaß}(\Gamma_2) \cong S_{2k-2}(SL_2(\mathbb{Z}))$ $F \leftrightarrow f$ then $Z_F(s) = \zeta(s-k+1)\zeta(s-k+2)L_f(s)$.

About $S_k^2(\Gamma_2)$ one knows almost nothing except for the dimension. Here is a table: (*)

We have to explain here the notions 'Klingen-Eisenstein-series' and 'Maaß-space'. The first one is as follows:

$$\text{Klingen-Eisenstein series} = \sum_{g \in C\backslash\Gamma_2} \tilde{f}|_k g, \quad \text{where } f \in M_k(\Gamma_1),\ \tilde{f}(\tau,z,\tau') := f(\tau),$$

C = centralizer of $\left(\begin{array}{c|c} 1 & 0\ * \\ \hline 0 & 1 \end{array}\right)$ in Γ_2.

The second one is where Jacobi forms enter the stage. So I should first explain what a Jacobi form is.

But before that a historical remark.

(*)

k	<20	20	22	24	26	28	30	—
$\dim S_k^2(\Gamma_1)$	0	1	1	2	2	3	4	—