The action of V_n^* on Fourier coefficients can be explicitly calculated. Thus it is a good idea to choose

$$\varphi = P_{n,r} := \begin{matrix}\text{n,r-th Poincaré series on } \Gamma_1^J \\ \text{of wt } k\end{matrix} = \sum_{\gamma \in \Gamma_{1,\infty}^J \backslash \Gamma_1^J} q^n \zeta^r |_{k,1}\, \gamma .$$

The characteristic property is

$$\langle \phi , P_{n,r} \rangle = (\text{const dep. only on } k) \times \begin{matrix}\text{n,r-th coefficient} \\ \text{of } \phi\end{matrix}$$

Thus

$$\langle \varphi_m | P_{n,r} | V_m \rangle = (\text{const. dep. only on } k) \times \begin{pmatrix}\text{n,r-th coefficient} \\ \text{of } \varphi_m | V_m^*\end{pmatrix}.$$

If we now insert ~~…~~ the formula for the n,r-th Fourier coefficients of $\varphi_m | V_m^*$ and if we assume for simplicity r^2-4n = fundamental discriminant we obtain

$$\cancel{\mathcal{D}_{F,P_{n,r}|V}(s) = \ldots \zeta(2s-2k+4) \sum_{\substack{\text{binary } Q \bmod SL_2(\mathbb{Z}) \\ \operatorname{disc} Q = r^2-4n}}}$$

$$\mathcal{D}_{F,P_{n,r}|V}(s) = \text{const} \sum_{\substack{[a,b,c] \bmod SL_2(\mathbb{Z}) \\ b^2-4ac = r^2-4n \\ \operatorname{disc} Q}} \zeta_Q(s) \sum_{n \geq 1}^{\infty} \frac{C(nQ)}{n^s}$$

where $F = \sum_{Q=[a,b,c]} C(Q)\, e^{2\pi i (a\tau + bz + c\tau')}$, $\zeta_Q = \sum_{\mathfrak{b} \sim \mathfrak{a}} \frac{1}{N(\mathfrak{b})^s}$, $\mathfrak{a}$ ideal corresponding to Q via the usual correspondence.

One the other hand side one has the Andrianov-identities:

$$A_\chi Z_F(s) = \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s} \sum_{\substack{Q \bmod SL_2(\mathbb{Z}) \\ \operatorname{disc} Q = r^2-4n}} \chi(Q) \sum \frac{C(nQ)}{n^s} \quad (\chi \text{ any ideal class character}).$$

Combining these two identities ~~and also~~ we get

<u>Theorem</u> Let F = Hecke-eigenform, G = Maass, then

$$\mathcal{D}_{F,G}(s) = \langle \varphi | \varphi_1 \rangle\, Z_F(s).$$