Now the Theorem follows from the Lemma:

Lemma ~~$E_s^*(Z)$ ...~~ $E_s(Z)$ has an analytic continuation to $\mathbb{C}$, the only poles being at $s=2$ of residue $\frac{\pi^2}{\zeta(4)}$. One has $E_s^*(Z) := \pi^{-s}\Gamma(s)\zeta(2s)E_s(Z)$ $= E_{2-s}^*(Z)$.

The proof is also fairly standard:
Using that $g \mapsto (0\,0\,0\,1)g$ induces a bijection $\mathcal{C}\backslash\Gamma_2 \xrightarrow{\sim} \{$primitive vectors in $\mathbb{Z}^4\}$ you get

~~$E_s^*(Z) = \int_0^\infty \Theta_t(Z) \ldots$~~

$$\zeta(2s)\, E_s(Z) = \sum_{\substack{c,d\in\mathbb{Z}^2\\ (c,d)\neq(0,0)}} \frac{|Y|^s}{Y[Z^*c+d]^s} \qquad \left(Z^* = \begin{bmatrix} \tau' & -z \\ -z & \tau \end{bmatrix}\right)$$

then write the right hand side as Mellin transform of a theta series and apply the usual machinery.

As mentioned above $D_{F,G}(s)$ behaves for even k exactly like the spinor zeta function attached to Siegel modular forms of ~~degree~~ 2, ~~weight~~ k.
So take e.g. a Hecke eigenform F, then $D_{F,F}(s)$ ~~behaves exactly like~~ has the same functional equation as $Z_F(s)$ = spinor zeta function of F.
~~except~~ But note: ~~if F~~ If $F \neq$ Maass then $Z_F(s)$ is holomorphic (Evdokimov)
whereas $\mathrm{res}_{s=k} D_{F,F} \approx \langle F,F\rangle \neq 0$.
So what is the connection between $D_{F,G}(s)$ and Andrianov zeta function.
Unfortunately Kohnen and myself cannot give a complete answer here, but only a particular result, which may be ~~~~ clearly enough.
In general we cannot say anything about the coefficient $\langle \varphi_m | \psi_m \rangle$ since we do not know anything about the φ_m's, ψ_m's.
But consider known the case

$$G \in M_k^{\text{Maass}}(\Gamma_2), \text{ i.e. } G = \sum_{m=1}^\infty \psi | V_m \, e(m\tau'), \quad \psi \in J_{k,1}.$$

Then

$$\langle \varphi_m | \psi_m \rangle = \langle \varphi_m | \psi | V_m \rangle = \langle \varphi_m | V_m^*, \psi \rangle.$$