Theorem (Maass, 1979)

There exist "canonical" operators $V_m : J_{k,1} \to J_{k,m}$ such that the map $\phi \to \phi|V := \sum_{m\geq 0} \phi|V_m \, e^{2\pi i m \tau'}$ defines a Hecke-equivariant embedding $J_{k,1} \hookrightarrow M_k(\Gamma_2)$.

The image of this embedding is called the "Maass-space".

The second result is much newer and is that what I am going to speak about.

Let

$$F, G \in M_k^{cusp}(\Gamma_2), \quad F = \sum_{m \geq 0} \phi_m \, e^{2\pi i m \tau'}, \quad G = \sum_{m\geq 0} \psi_m \, e^{2\pi i m \tau'}.$$

In perfect analogy with the Rankin convolution series in the theory of elliptic modular forms we set

$$D_{F,G}(s) = \zeta(2s-2k+4) \sum_{n \geq 1}^{\infty} \frac{\langle \phi_n | \psi_n \rangle}{n^s}$$

One has

Lemma: $D_{F,G}(s)$ converges for $\mathrm{Re}\, s > k+1$

$(*)$

Now, in analogy with the Rankin convolution of elliptic modular forms one would expect that $D_{F,G}(s)$ can be analytically continued, ~~and one of~~ has functional equation and so forth. And, indeed it has:

Theorem $D_{F,G}(s)$ has an analytic continuation to $\mathbb{C}$ with a possible pole only at $s=k$ ~~[illegible]~~ and with residue at $s=k$ equal to $\frac{2^{2k}\pi^{k+2}}{(k-1)!}\langle F|G\rangle$ ($\langle F|G\rangle = \int_{\Gamma_2 \backslash \mathfrak{H}_2} F \bar{G} \, |Y|^k \frac{dX\,dY}{|Y|^3}$ ($\begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} = X + iY$)). Moreover, it satisfies

$$D^*_{F,G}(s) := (2\pi)^{-2s}\,\Gamma(s)\,\Gamma(s-k+2)\,D_{F,G}(s) = D^*_{F,G}(2k-2-s).$$

Note that for even k the functional equation of $D_{F,G}(s)$ is identical with the functional equation satisfied by the spinor-zeta-function associated to elements of $M_k(\Gamma_2)$. Of course we have to investigate then what the connection between $D_{F,G}(s)$ and the Andrianov zeta function might be.