The official answer to this question is known and it runs as follows. One has

$$\mathcal{C} := \begin{array}{c}\text{centralizer of } \left[\begin{array}{c|c} 1 & \begin{smallmatrix}0&0\\0&1\end{smallmatrix} \\ 0 & 1\end{array}\right] \\ \text{in } \Gamma_2\end{array} = \left\{\begin{bmatrix} & * & & \\ 0&0&0&1\end{bmatrix}\right\} \cap \Gamma_2 \approx \Gamma_1 \ltimes H(\mathbb{Z}) =: \Gamma_1^J .$$

Here $H(\mathbb{Z})$ = Heisenberg group and $\Gamma_1 \ltimes H(\mathbb{Z})$ consists of triples $\gamma = A[\lambda,\mu]\kappa$ ($A \in \Gamma_1$, $\lambda,\mu,\kappa \in \mathbb{Z}$) with the multiplication law

$$\gamma\gamma' = AA'[(\lambda,\mu)A' + (\lambda',\mu')]\left(\kappa+\kappa' + \begin{vmatrix}(\lambda,\mu)A \\ (\lambda',\mu')\end{vmatrix}\right). \qquad (\gamma' = A'[\lambda',\mu']\kappa').$$

Γ_1^J acts on $\mathfrak{h}\times\mathbb{C}$ by $\gamma\cdot(\tau,z) = \left(A\tau, \frac{z+\lambda\tau+\mu}{c\tau+d}\right)$ and for each $k,m \in \mathbb{Z}$ ~~function on $\mathfrak{h}\times\mathbb{C}$~~ there exists an action on function on $\mathfrak{h}\times\mathbb{C}$:

$$(\phi|_{k,m}\gamma)(\tau,z) = \phi(\gamma\cdot(\tau,z))\,(c\tau+d)^{-k} e^{2\pi i m\left(\frac{-c(z+\lambda\tau+\mu)^2}{c\tau+d} + \lambda^2\tau + 2\lambda z + \kappa\right)} \quad \left(A = \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\right)$$

Now the ϕ_m belong to

$$J_{k,m} = \left\{\phi : \mathfrak{h}\times\mathbb{C} \xrightarrow{\text{hol.}} \mathbb{C} \;\middle|\; \begin{array}{l}\phi|_{k,m}\gamma = \phi \;\; \forall \gamma \in \Gamma_1^J \\ \phi \text{ ~~satisfies a certain reg.~~ is "regular at the cusp } \infty\text{"}\end{array}\right\} = \begin{array}{l}\text{space of} \\ \text{Jacobi forms} \\ \text{of index } m, \\ \text{weight } k.\end{array}$$

As I mentioned in the beginning there is a well-developed theory of Jacobi forms. Basic notions:

$J^{cusp}_{k,m}$ = subspace of cusp forms $\quad (F \in M_k^{cusp}(\Gamma_2) \Rightarrow \phi_m \in J^{cusp}_{k,m})$,

$$\langle\phi|\psi\rangle := \begin{array}{l}\text{Petersson} \\ \text{scalar} \\ \text{product of } \phi,\psi\end{array} = \int_{\Gamma_1^J\backslash\mathfrak{h}\times\mathbb{C}} \phi\overline{\psi}\, e^{-4\pi m\frac{y^2}{v}} v^k\, dV$$

($\phi,\psi \in J^{cusp}_{k,m}$; $y = \operatorname{Im} z$, $v = \operatorname{Im}\tau$, dV = Γ_1^J-invariant volume on $\mathfrak{h}\times\mathbb{C}$ such that $\int_{\Gamma_1^J\backslash\mathfrak{h}\times\mathbb{C}} dV = 1$)

Hecke operators $T(\ell)$ $(\ell \in \mathbb{Z}_{>0})$,

Main Theorem: $J_{k,m} \approx$ certain "canonical" ~~Hecke-equiv.~~ subspace of $M_{2k-2}(\Gamma_0(m))$.

Now let us go back to the Fourier-Jacobi-development of Siegel modular forms. In the direction of the above mentioned program there ~~are two~~ ~~in the very~~ ~~few~~ are essentially only two results so far I know. The first one is due to Maass: