For 2/ very little to calibrate!

Eis Hecks) As example let us consider the case $N = p^2$ prime

$\#$ cusps $= p+1$ $\qquad$ $\#$ Eisen Hecks $= p$

Here they are (recall that the line in $Hecke_{\Gamma_0}(\mathbb{Z}[P^1(\mathbb{Q})], \mathbb{Q})^d$ w.c.o.)

Fix χ pr. Dirichlet character mod p

$$\lambda\left(\left(\tfrac{p}{q}\right)\right) = \chi(p)\,\chi(q/p)$$

(and $\lambda\left(\tfrac{p}{q}\right) = \begin{cases} 1 & \ell \mid q \\ 0 & \text{o.w.} \end{cases} = \sum_{\ell \mid \mathrm{den}(\cdot)} \ell$ and $= \begin{cases} 1 & \ell^2 \mid q \\ 0 & \text{o.w.} \end{cases}$)

In fact $\lambda\left(\dfrac{ap + bq}{cp + dq}\right) = \chi(ap + bq)\,\chi\left(\dfrac{cp + dq}{\ell}\right)$

$\ell \mid cp + dq \iff \ell \mid q$, if $\ell \mid q$ then:

$$= \chi(ap)\,\chi(d)\,\chi\left(\tfrac{q}{\ell}\right) \text{ and } ad \equiv_{\ell^2} 1 \;.\; \square$$

Cheery exercise show that these are Hecks.

or factorial nonsense!

Twists by real quadr. character: $\lambda = \lambda_0 \otimes \left(\dfrac{\cdot}{\ell}\right)$

$$\lambda\left(\sum c_s\,(s)\right) := \lambda_0\left(\sum c_s \left(\tfrac{p}{\ell}\right)\left(\tfrac{q/\ell}{\ell}\right)(s)\right)_{s = \frac{p}{q}}$$

$f(z) \leftrightarrow f(dz)$ $\qquad$ $\lambda\left(\sum c_s(s)\right) := \lambda_0\left(\sum c_s\,(ds)\right)$