3/ What you actually would use for explicit calculations (but you loose the simplicity of the formulas):

$$\mathbb{Z}[\mathbb{P}^1(\mathbb{Q})]^0 \quad \text{is} \quad \mathbb{Z}[\Gamma]\text{-module of rank } 1 \qquad (\Gamma = SL(2,\mathbb{Z}))$$

generator e.g. $(\infty)-(0)$.

[pf. Use an idea of Manin:

$$\sum c_s (s) = \sum c_s((s)-(\infty))$$

$s = \frac{p}{q}$: $\quad \frac{p_{-1}}{q_{-1}} = \infty = \frac{1}{0}, \ \frac{p_0}{q_0} = [s], \ \ldots, \ \frac{p_n}{q_n} = \frac{p}{q}$

convergents of contin. fract. exp. of $\frac{p}{q}$

then $$(s)-(\infty) = s - \frac{p_{n-1}}{q_{n-1}} + \frac{p_{n-1}}{q_{n-1}} - \frac{p_{n-2}}{q_{n-2}} + (\ldots) + \frac{p_0}{q_0} - \frac{p_{-1}}{q_{-1}}$$

$$\begin{pmatrix} p_\nu & \pm p_{\nu-1} \\ q_\nu & \pm q_{\nu-1} \end{pmatrix}((\infty)-(0))$$

$\in SL(2,\mathbb{Z})$ if $\pm$ suit. chosen.]

You arrive at

Thm. M an obtain

$$\mathrm{Hom}_{\mathbb{Z}}\underbrace{(\mathbb{Z}[\Gamma_0(N)\backslash\Gamma], \mathbb{C})}_{} \longleftarrow M$$

but

$$\approx \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z}) \qquad \left(A \overset{\text{via}}{\leftrightarrow} [0:1]A\right)$$

These maps $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z}) \to \mathbb{C}$ are what all people arrive at if they consider mod. symbols (after all Manin did it also.)

Stated in our language:

Thm. Let $\phi: \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z}) \to \mathbb{C}$ a map such that:

$$\phi(x)+\phi(xS)=0 \qquad \phi(x)+\phi(xR)+\phi(xR^2)=0$$

$$(S: 0 \leftrightarrow \infty, \quad R: \infty \mapsto 0 \mapsto -1)$$

then

$$\lambda\left(\sum c_s (s)\right) = \sum_s c_s \sum_{\nu=0}^{n_s} \phi\left([q_\nu^{(s)} : (-1)^{\nu+1} q_{\nu-1}^{(s)}]\right)$$

($\frac{p_\nu^{(s)}}{q_\nu^{(s)}}$ the conv. of s) def. el. of M.

M is generated by all these. $\Longrightarrow$