2/ Find $\mathbb{T}$-module $\mathbb{M} \xleftrightarrow{\cong} S_k(N)$

$\mathbb{M}$: generators explicitly given by 'finite' formulas and $\mathbb{T}$-action

$\xleftrightarrow{\cong}$: must not at all expl.

Then you immediately have a basis for $S_k(N)$ in closed formulas:

<u>Thm</u> $\quad S_k(N) = \left\{ \sum_{n=1}^{\infty} \lambda(T(n)x)\, q^n \;\middle|\; \lambda \in \mathbb{M}^* \right\}$

[pf: $S_k(N)^*$ generated by: $a_\ell : f \mapsto a_\ell(f)$ (ℓ-th Fourier coeff.)

Let $\phi : S_k(N) \longrightarrow \mathbb{M}$

$\lambda \in X^*$ given, then $\lambda \circ \phi = \sum_{\ell \in M} c_\ell a_\ell$ suit. $c_\ell \in \mathbb{C}$, M suit. finite set $\subseteq \mathbb{N}$

$x \in X$ ~~let $x = \phi(f)$:~~ $x = \phi(f)$ suit. f

$$\lambda(T(n)x) = \lambda \circ \phi(T(n)f) = \sum_{\ell \in M} c_\ell\, a_\ell(T(n)f)$$

$$= \sum_{\ell \in M} c_\ell\, a_n(T(\ell)f) \quad \dots \quad \text{hence } `\supseteq`$$

Cruc. pt.: $\quad a_n(T(\ell)f) = a_\ell(T(n)f)$

You can reverse: f given, take $x := \phi(f)$, $\lambda : \lambda \circ \phi = a_1$. $\square$

of course:

<u>Variations</u>: sub or super spaces of $S_k(N) \xleftrightarrow[\mathbb{T}\text{-mod.}]{\cong}$ suit. X

<u>Possible choices</u> for $\mathbb{M}$: $\Longrightarrow$