INTRODUCTION

In the following article we consider holomorphic modular forms with respect to the congruence subgroup $\Gamma_0(N) = \{(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma : c \equiv 0 \bmod N\}$ of the modular group Γ. Of course the holomorphy condition includes the cusps. By $S_k(\Gamma_0(N),\chi)$ we denote the space of these forms of weight k and character χ, i.e. those satisfying

$$f(\frac{az + b}{cz + d})\ (cz + d)^{-k} = \chi(a)\ f(z)$$

for all substitutions of $\Gamma_0(N)$, which vanish in the cusps.

A

Our main problem, the basis problem, is to give bases of linearly independent forms of these spaces which are arithmetically distinguished and whose Fourier series are known or easy to obtain.

B

The solution rests on an arithmetic counterpart to Hecke's theory in the arithmetic of quaternion algebras. It has been initiated by Brandt, and we may reasonably call the analogues of Hecke's T(n) the Brandt matrices. The link between them and the T(n) is the fact that both generate isomorphic semisimple rings with the same traces. The determination of the traces is therefore our chief concern.

In I we give a proof of the fact that theta series (generalized by Schoeneberg) are modular forms. It differs from the classical proof by Hermite in the respect that the (finite) modular congruence group $\Gamma/\Gamma(N)$ is used in an essential way. We think that this procedure seems less artificial. In this connection we meet the problem of the representations of this group which is today only partly solved.

In II we develop the arithmetic of quaternion algebras, concluding with the computation of the traces of the Brandt matrices. In III the traces of the Hecke operators in the space of modular forms with respect to the congruence subgroups $\Gamma_0(N)$, with a square-free N, are