(6) There are other ~~ideas~~ solution to the basis problem which I want to describe, has its roots in ideas of Manin. The idea is essentially to describe modular forms via their periods. The key words for this theory are "Eichler – Shimura – isomorphism", modular symbols. I am starting with modular symbols too, but maybe in a little unusual way.
Let me start with some formal definitions.

(7) It is easy to deduce that the above pairing also induces a perfect pairing between

$$C^{\varepsilon}_{\mathbb{R}}(\Gamma)\ ,\ S_{\mathbb{R}}(\Gamma).$$

Hence $C^{\varepsilon}_{0\mathbb{R}}(\Gamma)^* \approx S_k(\Gamma)$ as Hecke-Modules.
I do not know what this isomorphism looks like, but it is easy to describe all Hecke-Homomorphisms:

<u>Theorem</u> Let $\sigma \in C^{\varepsilon}_{0\mathbb{R}}(\Gamma)$. Then the correspondence

$$\varphi \longmapsto \sum_{1}^{\infty} \varphi(T(l)\sigma)\, q^{l}$$

defines a Hecke-equivariant map

$$L_\sigma : C^{\varepsilon}_{0\mathbb{R}}(\Gamma)^* \longrightarrow S_k(\Gamma).$$

There is a σ s.th. L_σ = isom.

How does one define or describe explicitly such φ?
It would be possible now to give an ad-hoc construction, but it would not be clear where this comes from. So I ~~shall~~ develop more theory first.