2nd case: signature of $\circ$ = $1, n-1$

Consider

$$p(Z) = q(Z_\perp)\, e^{\cancel{2\pi v}\, Z_\perp^2} \qquad \text{(same notation as above).}$$

Then

$$\left(\frac{-1}{4\pi}\Delta + E\right) p(Z) = (1 - n - \deg q)\, p .$$

Thus

$$\vartheta_\circ(\tau, z) = v^{-\frac{n}{4}} \vartheta(\tau, z) = \sum_{Z \in \Gamma} q(Z_\perp)\, e^{2\pi v Z_\perp^2}\, e^{\pi i(\tau Z^2 + 2 z Z Z_\circ)}$$

satisfies

$$\vartheta_\circ \big|^*_{k,m} \eta = \vartheta_\circ \quad \text{for all } \eta \in \Gamma_1(\ell)^J \quad \text{with} \quad \begin{aligned} k &= \tfrac{n}{2} + \deg p \\ m &= \tfrac{1}{2} Z_\circ^2 \end{aligned}$$

Here

$$f\big|^*_{k,m} \eta\,(\tau, z) = \text{same as } f\big|_{k,m}\eta\,(\tau,z), \text{ but with } (c\tau+d)^k \text{ replaced by } |c\tau+d|\,\overline{(c\tau+d)}^{\,k-1} .$$

Moreover note that

$$(8\pi i m\, \partial_\tau - \partial_z^2)\, \vartheta_\circ = \underbrace{\sum q(Z_\perp)\, e^{\cdots} \left[8m(\pi i)^2(-Z_\perp^2 + Z^2) - (2\pi i\, Z\cdot Z_\circ)^2 \right]}_{=0}$$

We define accordingly

$J^*_{k,m}(\Gamma)$ = skew-holomorphic Jacobi forms of index m, weight k on Γ ($\subseteq Sl_2(\mathbb{Z})$ finite index)

$$= \left\{ \phi : \mathfrak{H} \times \mathbb{C} \xrightarrow{\text{smooth}} \mathbb{C} \;\middle|\; \begin{array}{l} \phi \text{ holomorphic in } z,\ (8\pi i m\, \partial_\tau - \partial_z^2)\phi = 0 \\ \phi\big|^*_{k,m} \eta = \phi \quad \forall \eta \in \Gamma^J \\ \text{For all } A \in Sl_2(\mathbb{Z}) \text{ one has a Fourier expansion of the form} \\ \phi\big|^*_{k,m} A = \sum_{r^2 - 4mn \geq 0} c(n,r)\, e\!\left(n\tau + \frac{r^2 - 4mn}{2m}\, i v + r z\right) \end{array} \right\}.$$

Note that the ~~regularity~~ condition $r^2 - 4mn \geq 0$ is obviously fullfilled by the $\vartheta_\circ$ above which serves here as prototype for skew-holomorphic forms.