More generally, one can associate theta series to indefinite quadratic forms which leads to non-holomorphic Jacobi forms:

Theorem

Let V be a real vector space of even dimension $n < \infty$, let $\cdot$ be a scalar product on V (not nec. pos. definite), let $p(x)$ be a real function on V such that:

(i) $p(x)\, e^{-\pi x^2} \in \mathcal{S}(V)$ (Schwartz space)

(ii) $$\left(-\frac{1}{4\pi}\Delta + E\right)p = \left(k - \frac{n}{2}\right)p$$ for some $k \in \mathbb{Z}$.

(Δ = Laplacian associated to $(V, \cdot)$, i.e. if $y_1, \dots, y_n$ are coordinates with respect to an orthogonal basis, so that $x^2 = y_1^2 + \dots + y_r^2 - y_1^2 - \dots - y_s^2$, then $\Delta p(x) = \left(\frac{\partial^2}{\partial y_1^2} + \dots + \frac{\partial^2}{\partial y_r^2}\right) p - \frac{\partial^2}{\partial y_1^2} - \dots - \frac{\partial^2}{\partial y_s^2} p$, $E = \sum y_i \frac{\partial}{\partial y_i}$)

~~Finally let $X_0 \in V$~~

Finally let $\Gamma \subseteq V$ be an even integral lattice, $Z_0 \in \Gamma$.

Then

$$\vartheta_p(\tau, z) := \vartheta^{\frac{n}{4} - \frac{k}{2}} \sum_{Z \in \Gamma} p\left(\sqrt{\vartheta}\left[Z + \frac{y}{\vartheta} Z_0\right]\right) e\left(\tau \frac{Z^2}{2} + z\, Z Z_0\right)$$

transforms under $\Gamma_1(\ell)$ (ℓ = level of Γ) like a Jacobi form of weight k and index $m = \frac{Z_0^2}{2}$.

Proof: Do not want to prove this. Mimic the proof of the corresponding Theorems for modular forms (i.e. $Z_0 = 0$) as given in Vignéras ...)

This yields examples of holomorphic and nonholomorphic Jacobi forms.

There are especially two interesting cases:

~~1st case: $\cdot$ = pos. definite~~

$\cdot$ = pos. def., $\frac{n}{2}, \frac{m}{2} Z_0^2 > 0$, $p(x) = q(X_\perp)$ where q = spherical polynomial of degree k on orthogonal complement of Z_0 in V and $X_\perp$ = orthogonal projection of X onto this complement.

This gives elements of

$\vartheta(\tau, z) \in J_{k,m}(\Gamma_1(\ell))$.