Proposition: The map $\phi = \sum h_g \vartheta_{m,g} \mapsto \sum h_g \otimes \vartheta_{m,g}$ yields an isomorphism

$$J_{k,m} \approx \left(M_{k-1/2}(\Gamma(4m)) \otimes \mathrm{Span}_{\mathbb{C}}\{\vartheta_{m,g} \mid g=1,\dots,2m\}\right)_{SL_2(\mathbb{Z})}.$$

This is easy, but it has great corollaries.

Corollary 1: $\dim J_{k,m} < \infty$ (actually $= \frac{km}{6} + O(1)$ for $k \to \infty$ as will be seen later)

and more generally

Corollary 2: Questions about Jacobi forms can be answered in terms of modular forms of half integral weight, e.g. traces of double coset operators acting on $J_{k,m}$ can be reduced to trace formulas for modular forms of half integral weight.

However in general it is better to forget this Corollary 2 and to try to remain within the theory of Jacobi forms, since Jacobi forms behave more natural and tend to give more canonical results than the theory of modular forms of half integral weight. In any case the above proposition explains in what sense one might view the theory of Jacobi forms as a generalisation of modular forms of half integral weight.

We want to discuss the last example of a Jacobi theta function more thoroughly by restricting to the smaller class of arithmetically defined Theta series.

3rd example: Jacobi Theta series

The following is known and gives an example of a Jacobi theta function:

$\Lambda \subseteq \mathbb{R}^n$ lattice, even, integral of level ℓ, $x_0 \in \Lambda$ fixed vector (n even), ~~then~~

$$\vartheta(\tau,z) = \sum_{x\in\Lambda} q^{x^2/2}\zeta^{x\cdot x_0} \in J_{\frac{n}{2},\frac{x_0^2}{2}}(\Gamma_1(\ell)).$$

(References are: Hecke, Schoeneberg, Kloosterman ...).

It is possible to generalise this. Namely one can prove the following Theorem.