Here the sum is over $r \in \mathbb{Z}$ and n runs through a subset of rational numbers, but with bounded denominators. Note that the crucial point is the condition "$r^2 - 4mn \leqq 0$". The φ_m above obviously satisfy this condition and this serves here as model for the general case. However, there are other reasons why this condition is natural, as we shall see later.

An easy but important Lemma about these Fourier expansions is

<u>Lemma</u> Let $\phi = \sum c(n,r) q^n \zeta^r \in J_{k,m}$ $(:= J_{k,m}(SL_2(\mathbb{Z})))$. Then $c(n,r) = c(n',r')$ for $r \equiv r' \ (2m)$ and $r^2 - 4mn = r'^2 - 4mn'$.

There is also a corresponding statement for $\Gamma \leq SL_2(\mathbb{Z})$, but we do not need it. The Lemma follows easily from the transformation law of ϕ with respect to $\mathbb{Z}^2 (\leq \Gamma_1^J)$.

Thus the Fourier expansion of ϕ can also be written as

$$\phi = \sum_{\substack{\Delta, r \in \mathbb{Z}, \Delta < 0 \\ \Delta \equiv r^2 (4m)}} C_\phi(\Delta, r)\, q^{\frac{r^2 - \Delta}{4m}} \zeta^r ,$$

with $C(\Delta, r)$ depending only on r modulo $2m$, Δ being a discriminant.

<u>2nd example: elliptic functions</u>

Any elliptic curve over $\mathbb{C}$ is isomorphic to $\mathbb{C}/\mathbb{Z}\tau + \mathbb{Z}$ for some $\tau \in \mathfrak{H}$. Every elliptic function on $\mathbb{C}/\mathbb{Z}\tau+\mathbb{Z}$ is determined (up to multiplication by a constant) by its Divisor. More ~~precisely~~ generally one has an exact sequence

$$1 \longrightarrow \begin{pmatrix} \text{trivial theta} \\ \text{functions on } \mathbb{C}/\mathbb{Z}\tau+\mathbb{Z}: \\ z \mapsto e^{\alpha z^2 + \beta z + \gamma} \end{pmatrix} \longrightarrow \begin{pmatrix} \text{theta functions} \\ \text{on } \mathbb{C}/\mathbb{Z}\tau+\mathbb{Z} \\ (\text{any } f)\text{" = elliptic} \\ \text{on } \mathbb{Z}\tau+\mathbb{Z} \end{pmatrix}^\times \longrightarrow \begin{matrix} \text{Divisors} \\ \text{on } \mathbb{C}/\mathbb{Z}\tau+\mathbb{Z} \end{matrix} \longrightarrow 1.$$

Now give for any $\tau \in \mathfrak{H}$ a Divisor D_τ on $\mathbb{C}/\mathbb{Z}\tau+\mathbb{Z}$ and consider the elliptic function $f(z) = f(\tau, z)$ corresponding to this divisor. If D_τ varies nicely with τ the $f(\tau,z)$ will be, or at least will behave like a Jacobi form. As an example let $D_\tau = N(\frac{1}{N}) - N(0)$, $f(z) = f(\tau, z)$ (such that that elliptic fct on $\mathbb{Z}\tau+\mathbb{Z}$) div $f(z) = D_\tau$ and $f = \frac{1}{z^N} + O(\frac{1}{z^{N-1}})$.

Then it is easily checked, immediately from the definition of f, that

$$\tilde{f}(\tau, z) := f\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau+d}\right)(c\tau+d)^{-N}$$