~~The precise statement of this Lemma really doesn't matter, but~~ Note that by the Lemma we immediately deduce, that the $\varphi_m(\tau, z)$ must satisfy a certain transformation law with respect to $Sl_2(\mathbb{R}) \ltimes HI(\mathbb{R})$. Before I write this down let me introduce the Jacobi group. Usually people consider not $Sl_2(\mathbb{R}) \ltimes HI(\mathbb{R})$ but most [illegible] the subgroup corresponding to $\left(\begin{array}{c|c} 1 & \begin{smallmatrix} 0 & 0 \\ 0 & * \end{smallmatrix} \\ \hline 0 & 1 \end{array}\right)$.

~~Definition~~ The Jacobi group is the group $J(\mathbb{R}) = Sl_2(\mathbb{R}) \ltimes \mathbb{R}^2 \cdot S^1$, where the multiplication law is given by

$$(A\,[\lambda,\mu]\,s)(A'\,[\lambda',\mu']\,s') = AA'\,[(\lambda,\mu)A' + (\lambda',\mu')]\;ss'\,e\left(\begin{vmatrix}\lambda & \mu \\ \lambda' & \mu'\end{vmatrix}\right).$$

(In this notation we tacitly identify $Sl_2(\mathbb{R})$, $\mathbb{R}^2$ and S^1 with its images in $J(\mathbb{R})$; but note that $\mathbb{R}^2$ is not a subgroup of $J(\mathbb{R})$.)

We[illegible] occasionally use

$$\Gamma^J := \Gamma \ltimes \mathbb{Z}^2 \quad (= \text{obvious subgroup in } J(\mathbb{R})) \quad \text{for } \Gamma \leq Sl_2(\mathbb{Z}).$$

The Jacobi group acts on $\mathfrak{H} \times \mathbb{C}$, $\mathfrak{H}$ = Poincaré-upper half plane by

$$A\,[\lambda,\mu]\,s\;(\tau, z) = \left(\frac{a\tau+b}{c\tau+d}, \frac{z+\lambda\tau+\mu}{c\tau+d}\right). \qquad ((\tau,z) \in \mathfrak{H} \times \mathbb{C})$$

Compare this ~~to~~ the Lemma. Finally it acts on functions φ on $\mathfrak{H} \times \mathbb{C}$ by

$$(\varphi|_{k,m} A)(\tau,z) = \varphi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) e^m\left(\frac{-cz^2}{c\tau+d}\right)(c\tau+d)^{-k}$$

$$(\varphi|_{k,m}[\lambda,\mu]s)(\tau,z) = \varphi(\tau, z+\lambda\tau+\mu)\, e^m(\lambda^2\tau + 2\lambda z + \lambda\mu)\, s^m.$$

It is now easy to deduce from the Lemma and using these notations

$$F \in M_k(\Gamma_2) \implies \varphi_m|_{k,m}\,\gamma = \varphi_m \quad \text{for all } \gamma \in \Gamma_1^J.$$

Of course this generalizes in an obvious way to congruence subgroups of Γ_2.

We define accordingly for $\Gamma \underset{\text{fin. index}}{\leq} Sl_2(\mathbb{Z})$:

$$J_{k,m}(\Gamma) = \text{space of Jacobi forms of weight } k, \text{ index } m$$

$$= \left\{ \varphi : \mathfrak{H} \times \mathbb{C} \xrightarrow{\text{holomorphic}} \mathbb{C} \;\middle|\; \begin{array}{l} \varphi|_{k,m}\gamma = \varphi \quad \text{for all } \gamma \in \Gamma^J, \\ \text{for each } A \in Sl_2(\mathbb{Z}) \text{ one has a Fourier development of the form:} \\ \varphi|_{k,m} A\,(\tau,z) = \sum_{r^2-4mn \leq 0} c_A(n,r)\, q^n \zeta^r \quad (q = e^{2\pi i\tau},\ \zeta = e^{2\pi i z}) \end{array} \right\}$$