Thm Let $0 < s_1 < s_2 < \dots < s_\ell$ be integers, set $a_{ij} = s_j - s_i$ $(i<j)$. Then

$$\prod_{i<j} \vartheta_{a_{ij}} \Big/ \eta^{\binom{\ell-1}{2}}$$

is a proper Jacobi form.

$x < y < z$

$\overset{a}{y-x}$ $\quad \underset{c}{z-x}$

$\underset{b}{z-y}$

Ex $\ell = 3$/ quark-series

$\ell=5$/ $10\ \vartheta$'s$/\eta^6$ (see the wt 2-examples)

proof Set $x_i = s_i x$. Suffice to prove

$$\sum_{i<j} B(x_j - x_i) \ge \binom{\ell-1}{2}/24 \qquad \text{for all } x_i, x_j. \quad (**)$$

Adv. exercise. □

Work in progress:

Let $\Lambda \subseteq \mathbb{R}^\ell$ be a root system, then ((e_i) = standard basis of $\mathbb{R}^\ell$)

$$\sum_{\alpha\in\Lambda} B\Big(\sum_i (\alpha, e_i)\, x_i\Big) \ge \frac{\#(\Lambda)}{24} \qquad \text{for all } x_1, \dots, x_\ell \in \mathbb{R}.$$

$(**)$ is the case A_4.

Very likely interpretation in terms of Weyl-Kac-den. formula for affine Kac-Moody algebras.

Thm $$\Theta_{a,b} = \sum_{r,s\in\mathbb{Z}} \left(\frac{s}{3}\right) q^{(r^2+rs+s^2)/3}\, \zeta^{(a-b)r + as}$$

proof Use $\eta = \sum_{n\in\mathbb{Z}} \left(\frac{12}{n}\right) q^{n^2/24}$ □

$$A_n = \{\pm(e_i - e_j) \mid 1 \le i < j \le n+1\} \subseteq \Big(\sum_i e_i\Big)^\perp \subseteq \mathbb{R}^{n+1}$$