what define Jacobi forms of small weight:

Thm. The map $f \mapsto \mathrm{ord}(f; x)$ defines an isomorphism

$$\nabla\mathcal{B} \xrightarrow{\cong} \left.\begin{array}{l}\text{group of functions (add.)}\\ \text{generated by the } \mathcal{B}(ax)\ (a>0)\\ \text{and } \frac{1}{24}\ .\end{array}\right\} =: W$$

proof: let $f = \eta^{c_0} \prod_{\ell \geq 1} \vartheta_\ell^{c_\ell}$

$\mathrm{ord}(f;x) = \frac{c_0}{24} + \sum_{\ell \geq 1} c_\ell \, \mathcal{B}(\ell x)$

$$\mathcal{B}(x) = \frac{1}{4\pi^2} \sum \frac{e^{2\pi i n x}}{n^2} + \frac{1}{24}$$

$$\mathrm{ord}(f;x) = \frac{1}{4\pi^2} \sum \frac{p(e^{2\pi i n x})}{n^2}$$

$$p(t) = \sum_{\ell \geq 1} c_\ell t^\ell + \sum_{\ell \geq 0} c_\ell$$

$\mathcal{B} \leftrightarrow p$ defines isom. $W \xrightarrow{\cong} \mathbb{Z}[t]$

$\mathbb{Z}[t] \ni p \leftrightarrow \left(\prod_{\ell \geq 1} \vartheta_\ell^{c_\ell}\right) \eta^{2a_0 - p(1)}$ defines $\mathbb{Z}[t] \xrightarrow{\cong} \nabla\mathcal{B}$ $\square$

In terms of the isomorphism

$$\begin{array}{ccc}\mathbb{Z}[t] & \longrightarrow & \nabla\mathcal{B},\\ p & \longmapsto & f\end{array} \quad p = \sum a_\ell t^\ell \to \eta^{2a_0 - p(1)} \prod_{\ell \geq 1} \vartheta_\ell^{a_\ell}$$

we find

$$\begin{cases} f \text{ hol. in } \mathfrak{H} \times \mathbb{C} & \text{iff } \frac{1}{N} \sum_{\zeta^N = 1} p(\zeta) \geq p(0) \quad \forall N \geq 1 \\ f \text{ hol. at } \infty & \text{iff } \sum_{n \neq 0} \frac{p(e^{2\pi i n x})}{n^2} \geq 0 \quad \forall x \\ \mathrm{wt}\, f & = \frac{1}{2} p(0) \end{cases}$$