the ϑ_a's or ϑ's we need ~~a~~ notion of order at ∞.
for weak Jacobi forms.
Here it is:

<u>Def</u> For $\phi \in J^{weak}_{k,m}$ (char) set

$$\mathrm{ord}_\phi(x) := \min_{\substack{n,r \\ c_\phi(n,r)\neq 0}} mx^2 + rx + n$$

Then

(1) $\mathrm{ord}_\phi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$

(2) $\mathrm{ord}_\phi(x) \geq 0$ iff ϕ is ~~proper~~ Jacobi form (i.e. not weak)

(3) $\mathrm{ord}_{\phi\cdot\psi} = \mathrm{ord}_\phi + \mathrm{ord}_\psi$

(4) $\mathrm{ord}_{\phi|U_d}(x) = \mathrm{ord}_\phi(dx)$

(5) $\mathrm{ord}_\vartheta(x) = B(x)$, $B(x) = \frac{1}{2}(y-\frac{1}{2})^2 \begin{cases} 0 \leq y < 1 \\ x \equiv y\ (\mathbb{Z}) \end{cases}$

For (5): $\displaystyle \mathrm{ord}_\vartheta(x) = \min_{\substack{r\in\mathbb{Z} \\ \text{odd}}} \frac{x^2}{2} + \frac{r}{2}x + \frac{r^2}{8} \left(= \frac{1}{2}\left(x+\frac{r}{2}\right)^2\right)$

For (1)–(4): Follows essentially from $|\phi|e^{-2\pi m x^2 v}(\tau, x\tau+\mu) \sim e^{-2\pi v\, \mathrm{ord}(\phi,x)}$ $v \geq L(\tau) \to \infty$

$$= \left|\sum c_\phi(n,r)\, e^{2\pi i(nu + \dots)} e^{-2\pi v(n + rx + mx^2)}\right|$$

and $\phi \in J^{hol}_{k,m}$ iff $\phi\, e^{-2\pi m x^2 v} = O(1)$ for $v \to \infty$.

~~Hence $\sum S_{a_i}$ … $\in J^{hol}_{k,m}$ (char) iff $\min_x \sum \mathrm{ord}_{S_{a_i}}(x) = \frac{1}{24}$.~~ $\square$