Simplest Jacobi form:

Nils-Peter Skoruppa

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Factorization of theta series and lattices

$$\vartheta(\tau,z) = \sum_{r\in\mathbb{Z}} \left(\frac{-4}{r}\right) q^{r^2/8}\, \zeta^{r/2}$$

$$= q^{1/8}(\zeta^{1/2}-\zeta^{-1/2}) \prod_{n\geq 1}(1-q^n)(1-q^n\zeta)(1-q^n\zeta^{-1})$$

Zeros for fixed z: $\mathbb{Z}\tau+\mathbb{Z}$

$\vartheta(\tau,z) = \eta^3 \times$ Weierstraß σ-fct. ass. to $\mathbb{Z}\tau+\mathbb{Z}$.

$= \eta^3 z + O(z^3)$

$\vartheta \in J_{\frac12,\frac12}(\varepsilon^3\ \cancel{\nu_\eta^3 \times \nu_H})$

$Mp(2,\mathbb{Z})^J := Mp(2,\mathbb{Z}) \ltimes \mathbb{Z}^2\cdot\mathbb{Z}$

Thm

The 1-dim characters of $Mp(2,\mathbb{Z})^J$ are

$$(A,w)(\lambda,\mu)x \longmapsto \varepsilon(A,w)^a \underbrace{(-1)^{(\lambda+\mu+\lambda\mu+x)b}}_{\nu_H(\ )} \quad \left(\begin{matrix} a \bmod 24 \\ b \bmod 2\end{matrix}\right)$$

where $\varepsilon\ \cancel{\nu_\eta}(A,w) = \dfrac{\eta(A\tau)\, w(\tau)^{-1}}{\eta(\tau)}$ $\quad \left(\eta(\tau) = q^{\frac{1}{24}}\prod_{n\geq 1}(1-q^n)\right)$.

$k,m\in\frac12\mathbb{Z}$: ε^a

(*) $J_{k,m}(\varepsilon^a) := \{\phi : \mathfrak{H}\times\mathbb{C} \xrightarrow{hol} \mathbb{C} \mid$ i) $\phi|_{k,m}(A,w) = \phi(A\tau, \frac{z}{c\tau+d})\, w(\tau)^{-2k}\, e^m(-cz^2/c\tau+d) = \varepsilon^a(A,w)\,\phi$

iii) $\phi = \sum_{\substack{4nm-r^2\geq 0\\ n,r\in\mathbb{Q}}} c_\phi(n,r)\, q^n \zeta^r$

ii) $\phi|_{k,m}[\lambda,\mu]x = \phi(\tau, z+\lambda\tau+\mu)\, e^m(\lambda^2\tau+2\lambda z+\lambda\mu+x) = \nu_H^b(\lambda,\mu,x)\,\phi$ $\}$

$b = \begin{cases} 0 & m\in\mathbb{Z} \\ 1 & m\in\frac12+\mathbb{Z}\end{cases}$

Note: ① Right hand side of (*) $\neq 0$

~~[illegible]~~

~~$\vartheta \in J_{1/2,1/2}(\nu_\eta^a \times \nu_H^b) \Rightarrow \vartheta(\tau,\ell z) \in J_{\dots}(\nu_\eta^a \times \nu_H^{b\ell})$~~

~~$\phi_j \in J_{k_j,m_j}(\nu_\eta^{a_j} \times \nu_H^{b_j}) \Rightarrow \prod_j \phi_j \in J_{\sum k_j, \sum m_j}(\nu_\eta^{\sum a_j} \times \nu_H^{\sum b_j})$~~

② $\bigoplus_{\substack{k\in\mathbb{Z}\\ m\in\frac12\mathbb{Z}\\ a \bmod 24}} J_{k,m}(\varepsilon^a)$ tri-graded algebra

③ $\phi(\tau,z)\longmapsto\phi(\tau,dz)$ def. $U_d : J_{k,m}(\varepsilon^a) \longrightarrow J_{k,md^2}(\varepsilon^a)$