(mod 2). Then $\mathfrak{M}_4'$ is defined in $\mathfrak{M}_4$, by $\sqrt{\chi_{68}|_{\mathfrak{M}_4}}$ (Tsuyumine [55]). It is not known if there is Siegel modular form of weight 34 whose restriction to $\mathfrak{M}_4$ equals $\sqrt{\chi_{68}|_{\mathfrak{M}_4}}$. If this is the case, $\mathfrak{M}_4'\subset\mathfrak{M}_4$ is in the case $D(1)$. However even if we are not lucky, we have only the case $D(2)$. Here we pose the following question:

Question. Is there a Siegel modular form f for Γ_4 such that the ideal (J_8, χ_{68}, f) defines R set-theoretically?

Suppose that the answer is affirmative. Then it can be shown that $R\subset\mathfrak{M}_4'$ satisfies the Assumption II, and hence that the above sequence if inclusions satisfies the conditions mentioned at the end of § 2. The result in the preceding section will help to determine the graded ring associated with R which corresponds to the image of the map;

$$H_3\times H\longrightarrow H_4$$
$$(Z_1, z_4)\longrightarrow\begin{bmatrix} Z_1 & 0 \\ 0 & z_4\end{bmatrix}.$$

So there will be a chance we get some information about the structure of $A(\Gamma_4)$.

We introduce two facts concerning the above question. In his paper [31], Igusa constructed the modular form of weight 540 which defines $R\cup\mathscr{H}_4$ in $\mathfrak{M}_4'$ set-theoretically, $\mathscr{H}_4$ denoting the hyperelliptic locus. Sasaki [38] gave for an arbitrary degree, a system of modular forms which defines set-theoretically the reducible locus. However its number is not small in general.

Though without evidence, the author believes that there should exist such a modular form as in the question. However it is hard to manage the step "$R\subset\mathfrak{M}_4'$" if one follows the method in § 5. We shall need to use it jointly with some other method, e.g., an asymptotic dimension formula for spaces of cusp forms for Γ_4 (up to the first or the second degree terms? $\cdots$, I don't know), of course, the perfect one is better. Or is it possible to attack directly the graded ring associated with $\mathfrak{M}_4'$ by geometric method? In this section we have shown only that the best card is still in a pile.

References

[1] Baily, W. L. and Borel, A., Compactification of arithmetic quotients of bounded symmetric domains, Ann. Math. (2), **84** (1966), 442–528.

[2] Blumenthal, O., Über Modulfunktionen von mehreren Veränderlichen, (Erste Hälfte), Math. Ann., **56** (1903), 509–548; (Zweite Hälfte), *ibid.*, **58** (1904), 497–527.

[3] Busam, R., Eine Verallgemeinerung gewisser Dimensionsformeln von Shimizu, Invent. Math., **11** (1970), 110–149.

[4] Cohn, L., The dimension of spaces of automorphic forms on a certain two dimensional complex domain, Mem. Amer. Math. Soc., **158** (1975).

[5] Fomenko, O. M., Modular forms and Hilbert functions for the field $Q(\sqrt{2})$ (Russian), Math. Zametki, **4** (1968), 129–136; (English transl.) Math. Notes, **4** (1968), 567–571.

[6] Freitag, E., Zur Theorie der Modulformen zweiten Grades, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II, (1965), 151–157.

[7] ——, Modulformen zweiten Grades zur rationalen und Gaußschen Zahlkörper, Sitz. der Heidelberg Akad. Wiss. No. 1, **49** (1967).

[8] ——, Stabile Modulformen, Math. Ann., **230** (1977), 197–211.

[9] ——, Die Irreduzibilität der Schottkyrelation (Bemerkung zu einem Satz von J. Igusa), Arch. Math., **40** (1983), 255–259.

[10] ——, Siegelsche Modulfunktionen, Grundlehren, **254**, Springer-Verlag, 1983.

[11] Van der Geer, G., Hilbert modular forms for the field $Q(\sqrt{6})$, Math. Ann., **233** (1978), 163–179.

[12] Van der Geer, G. and Zagier, D., The Hilbert modular group for the field $Q(\sqrt{13})$, Invent. Math., **42** (1977), 93–133.

[13] Gundlach, K.-B., Die Bestimmung der Funktionen zur Hilbertschen Modulgruppe des Zahlkörpers $Q(\sqrt{5})$, Math. Ann., **152** (1963), 226–256.

[14] ——, Die Bestimmung der Funktionen zu einigen Hilbertschen Modulgruppen, J. Reine Angew. Math., **220** (1965) 109–153.

[15] Hammond, W. F., On the graded ring of Siegel modular forms of genus two, Amer. J. Math., **87** (1965), 502–506.

[16] ——, The modular groups of Hilbert and Siegel, Amer. J. Math., **85** (1966), 497–516.

[17] Hermann, C. F., Symmetrische Hilbertsche Modulformen und Modulfunktionen zu $Q(\sqrt{17})$, Math. Ann., **256** (1981), 191–197.

[18] ——, Thetafunktionen und symmetrische Hilbertsche Modulformen zu $Q(\sqrt{65})$, J. Reine Angew. Math., **339** (1983), 147–162.

[19] Hirzebruch, F., Hilbert modular surfaces, L'Enseigment Math. (2), **21** (1973).

[20] ——, Kurven auf den Hilbertschen Modulflächen und Klassenzahlrelationen, Lect. Notes in Math., **412**, 75–93, Springer-Verlag, 1974.

[21] ——, Hilbert's modular group for the field $Q(\sqrt{5})$ and the cubic diagonal surface of Clebsch and Klein (Russian), Uspekhi Mat. Nauk, **31:5** (1976), 153–166; (English transl.) Russian Math. Surveys, **31:5** (1976), 96–110.

[22] ——, The ring of Hilbert modular forms for real quadratic fields in small discriminant, In: Modular functions of one variable, IV, Lect. Notes in Math., **627**, 287–323, Springer-Verlag, 1977.

[23] ——, Überlagerungen der projektiven Ebene und Hilbertsche Modulflächen, L'Enseigment Math., **24** (1978), 63–78.

[24] ——, Modulflächen und Modulkurven zur symmetrischen Hilbertschen Modulgruppe, Ann. Sci. École Nor. Sup. (4), **11** (1978), 101–165.

[25] Hirzebruch, F. and Van de Ven, A., Hilbert modular surfaces and the classification of algebraic surfaces, Invent. Math., **23** (1974), 1–29.

[26] Hirzebruch, F. and Zagier, D., Intersection number of curves on Hilbert modular surfaces and modular forms of Neben type, Invent. Math., **36** (1976), 57–113.

[27] ——, Classification of Hilbert modular surfaces, In: Complex analysis and algebraic geometry, 43–77, Iwanami Shoten, Tokyo, 1977.

[28] Igusa, J., On Siegel modular forms of genus two, Amer. J. Math., **84** (1962), 175–200.