genus two and of an elliptic curve, which is, for instance, written in terms of $\xi_1, \cdots, \xi_8$, as

$$(\xi_1, \cdots, \xi_5, t+\xi_6, t+\xi_7, t+\xi_8), \qquad t \longrightarrow \infty.$$

Let $S(2, 8)_0$ be the subring of $S(2, 8)$ which is generated by homogeneous elements $I(\xi_1, \cdots, \xi_8)$ of degree s, $s \geq 0$ such that

$$\lim_{t\to\infty} t^{-7s/3} I(\xi_1, \cdots, \xi_5, t+\xi_6, t+\xi_7, t+\xi_8)$$

makes sense. Then the image of $A'(\Gamma_3)$ by ρ_3 is contained in $S(2, 8)_0$. We obtain a homomorphism of graded rings

$$\Psi: S(2, 8)_0 \longrightarrow S(6) \otimes S(2, 4)$$

by using the above limit. There is a commutative diagram

$$\begin{array}{ccc} A'(\Gamma_3) & \xrightarrow{\quad \rho_3 \quad} & S(2, 8) \\ \Psi \downarrow & & \Psi \downarrow \\ A(\Gamma_2) \otimes A(\Gamma_1) & \xrightarrow{\rho_2 \otimes \rho_1} & S(6) \otimes S(2, 4) \end{array}$$

The Ψ of right hand side is a map given by polynomial calculation, and is, although sometimes not very easy, surely computable. Thus since $\rho_2 \otimes \rho_1$ is injective, the Ψ of left hand side is computable (at least in principle).

Let $A(i)$ be the ideal of $A(\Gamma_3)/(\chi_{18})$ generated by the modular forms f with $\nu(f) \geq i$, and let $\bar{A}(i) = A(i)/A(i+1)$. Here we note that $A(i)$ is defined differently from [56] (the present $A(i)$ equals $A(i)/(\chi_{18})$ in [56]), but $\bar{A}(i)$'s are the same except for what we mentioned in the above remark. $\bar{A}(i)$'s are $\bar{A}(0)$-modules. We fix three modular forms β, γ, δ with $\nu(\beta)=1$, $\nu(\gamma)=2$, $\nu(\delta)=3$ respectively. If $f \in A(\Gamma_3)$ is of order $i \equiv 0 \pmod 4$ (resp. $1, 2, 3 \pmod 4$), then $f/\chi_{28}^{i/4}$ (resp. $f\delta/\chi_{28}^{(i+3)/4}$, $f\gamma/\chi_{28}^{(i+2)/4}$, $f\beta/\chi_{28}^{(i+1)/4}$) is contained in $A'(\Gamma_3)$ and hence the Ψ-image is defined. For $i \equiv 0 \pmod 4$ (resp. $1, 2, 3 \pmod 4$) we denote by $\Psi(i)$ the map given by

$$f \longrightarrow \Psi(f/\chi_{28}^{i/4}) \ (\text{resp. } \Psi(f\delta/\chi_{28}^{(i+3)/4}), \Psi(f\gamma/\chi_{28}^{(i+2)/4}), \Psi(f\beta/\chi_{28}^{(i+1)/4})),$$

where in [56], some specific modular forms are taken as β, γ, δ. $\Psi(i)$ induces an injective homomorphism

$$\bar{A}(i) \longrightarrow \bar{A}',$$

which we denote also by $\Psi(i)$. As mentioned above, this map is computable. Let us identify $\bar{A}(i)$ with its $\Psi(i)$-image. Then it is shown that

$$\bar{A}(i) = \bar{A}(i+4) \qquad \text{for } i \geq 3,$$

and so that

$$A(\Gamma_3)/(\chi_{18}) \simeq \bar{A}(0) \oplus \bar{A}(1) \oplus \bar{A}(2) \oplus \bigoplus_{\mu=0}^{\infty} (\bar{A}(3) \oplus \bar{A}(4) \oplus \bar{A}(5) \oplus \bar{A}(6)) \chi_{28}^{\mu}$$

as modules over a graded subring of $\bar{A}(0)$ isomorphic to a polynomial ring. The structure of $\bar{A}(i)$, $i \leq 6$, could be determined, and hence some of the structure of $A(\Gamma_3)$ too. We refer the reader to [56, pp. 831–2] for the theorem about the structure of $A(\Gamma_3)$. Here we note that *the problems about a minimal system of generators, relations among them, are untouched yet as well as the rationality problem of the variety H_3/Γ_3 which seems one of the outstanding problems.*

We shall close this section with Taniyama's words ([50, Letter to M. Sugiura]), which appears in context, to talk about the theory of complex multiplication. Perhaps the author takes them in a wider sense than Taniyama's original view, which, however, seems still true: When Siegel modular functions can be handled as "easily" as elliptic functions, number theory will have developed in many directions.

§ 6. Siegel modular forms of degree four

Let C be a non-hyperelliptic curve of genus four. We identify C with its canonical curve in P^3. Then C is a complete intersection of a quadric and a cubic. There are two types of quadrics in P^3, namely, smooth quadrics and quadric cones. A curve C exhibited as a complete intersection of a quadric cone and a cubic is said to *have a vanishing theta constant* because one of the theta constants with even characteristics vanishes at the jacobian point corresponding to C if and only if C is such a curve. Let $\mathfrak{M}_4'$ be the moduli of such curves.

Let R denote the irreducible subvariety of the reducible locus of $\mathscr{A}_4$ corresponding to the direct products of 3-dimensional abelian varieties and elliptic curves, and let $\bar{\mathfrak{M}}_4$, $\bar{\mathfrak{M}}_4'$ be the closures in $\mathscr{A}_4$ of $\mathfrak{M}_4$, $\mathfrak{M}_4'$. There is a sequence of inclusions;

$$R \subset \bar{\mathfrak{M}}_4' \subset \bar{\mathfrak{M}}_4 \subset \mathscr{A}_4.$$

$\bar{\mathfrak{M}}_4$ is an irreducible divisor of $\mathscr{A}_4$, and is in the case $D(1)$, indeed it is defined by the single modular form J_8 of weight eight which is called *the Schottky invariant.* It is Schottky's theorem, however whose rigorous proof has been first given by Igusa [31]. Let χ_{68} be the modular form for Γ_4 defined to be the product of all theta constants with even characteristics