$$\mathscr{H}_3 = \operatorname{div}(\chi_{18}).$$

We have an exact sequence

$$0 \longrightarrow (\chi_{18}) \longrightarrow A(\Gamma_3) \xrightarrow{\rho_3} S(2, 8)$$

where for the structure of $S(2, 8)$, we refer the reader to Shioda [46].

Let R denote the reducible locus of $\mathscr{A}_3$. Then we have a sequence of inclusions

$$R \subset \mathscr{H}_3 \subset \mathscr{A}_3$$

where $\mathscr{H}_3$ (resp. R) is irreducible of codimension one in $\mathscr{A}_3$ (resp. $\mathscr{H}_3$). The sequence satisfies the conditions which are mentioned at the end of § 2. R corresponds to the subset

$$R_0 := \left\{ \begin{bmatrix} Z_1 & 0 \\ 0 & z_3 \end{bmatrix} \in H_3 \,\middle|\, Z_1 \in H_2,\ z_3 \in H \right\}$$

of H_3. The stabilizer subgroup of Γ_3 at R_0 is the image of

$$\Gamma_2 \times \Gamma_1 \longrightarrow \Gamma_3$$

$$\left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\} \longrightarrow \begin{bmatrix} A & 0 & B & 0 \\ 0 & a & 0 & b \\ C & 0 & D & 0 \\ 0 & c & 0 & d \end{bmatrix}.$$

The subset of R_0 consisting of diagonal matrices, is stable under the matrix $\begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \in \Gamma_3$ with

$$U = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The graded ring associated with R, which we denote by $\bar{A}(0)$, is given by

$$\left\{ \sum \psi \otimes j \in A(\Gamma_2) \otimes A(\Gamma_1) \,\middle|\, \sum \psi \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} j(z_3) \text{ is symmetric in } z_1, z_2, z_3 \right\}$$

where for two graded modules $M = \oplus_k M_k$, $N = \oplus_k N_k$, $M \otimes N$ always denotes a graded module $\oplus_k M_k \otimes N_k$. $\bar{A}(0)$ is easily understood because the structures of $A(\Gamma_1)$, $A(\Gamma_2)$ are known. So if the trick mentioned in § 2 is applicable to $R \subset \mathscr{H}_3$ and to $\mathscr{H}_3 \subset \mathscr{A}_3$, then we get some information about the structure of $A(\Gamma_3)$. It has been done in [56], whose sketch is given in the following (see also [57]).

$\mathscr{H}_3 \subset \mathscr{A}_3$ is in the case $D(1)$, indeed $\mathscr{H}_3$ is defined by χ_{18} as stated above. So the problem is reduced to determine the graded ring $A(\Gamma_3)/(\chi_{18})$, whose projective spectrum equals $\mathscr{H}_3$. Let us denote by $\nu(f)$, $f \in A(\Gamma_3)$, the vanishing order of $f|_{\mathscr{H}_3}$ at R. There is a cusp form χ_{28} of weight 28 whose restriction to $\mathscr{H}_3$ vanishes only at R, where for the definition of χ_{28} we refer the reader to [56]. χ_{28} is the modular form of lowest weight satisfying such a property, and $\nu(\chi_{28}) = 4$. Hence $R \subset \mathscr{H}_3$ is in the case $D(4)$.

Remark. In [56], we have defined the order of vanishing to be twice as much as in the present paper in order to unify notations. So the vanishing order $\nu(\chi_{28})$ has been written as 8 there.

$\mathscr{H}_3 \subset \mathscr{A}_3$ is corresponding to the union of loci in H_3 of theta constants $\theta\begin{bmatrix} u \\ v \end{bmatrix}$, $\begin{pmatrix} u \\ v \end{pmatrix}$ being even. There are just six theta constants which vanish identically on $R_0 \subset H_3$, e.g., $\theta\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}(Z)$. Let V be the irreducible component of the analytic subvariety in H_3 defined by $\theta\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}(Z) = 0$ which contains R_0. Let f be a modular form for Γ_3. Suppose that $\nu(f) \equiv 0 \pmod 4$. Then $(f/\chi_{28}^{\nu(f)/4})|_{V - \Gamma_3 R_0}$ is holomorphic since χ_{28} vanishes nowhere on $V - \Gamma_3 R_0$, and it extends holomorphically to V by Riemann's removable singularity theorem. Thus

$$\Psi(f/\chi_{28}^{\nu(f)/4})(Z_1, z_3) = \lim_{\substack{Z \to Z_0 \\ Z \in V}} (f/\chi_{28}^{\nu(f)/4})(Z), \qquad Z_0 = \begin{bmatrix} Z_1 & 0 \\ 0 & z_3 \end{bmatrix} \in R_0,$$

is well defined. Let Γ_2' be the subgroup in Γ_2 of index six which leaves a theta characteristic $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ stable mod 2. Then $\Psi(f/\chi_{28}^{\nu(f)/4})$ is shown to be contained in $\bar{A}' := \left\{ \sum \psi \otimes j \in A(\Gamma_2') \otimes A(\Gamma_1) \,\middle|\, \sum \psi \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} j(z_3) \text{ is symmetric in } z_1, z_3 \right\}$. The homomorphism Ψ and the graded ring $\bar{A}'$ can be constructed as the one which we introduced in § 2 where $X = \mathscr{H}_3$, and $D = R$.

Let F be a meromorphic modular form for Γ_3 whose restriction to V is, however, holomorphic such as $f/\chi_{28}^{\nu(f)/4}$ above. It is not easy to calculate $\Psi(F)$ directly from definition, particularly if F is not holomorphic globally on H_3. $\rho_3(F)$ is obviously well-defined and is contained in $S(2, 8) \subset \mathbf{C}[\xi_1, \cdots, \xi_8]$. Let $A'(\Gamma_3)$ be the graded ring of such modular forms. A hyperelliptic point Z in V moves toward $Z_0 \in R_0$ if the corresponding curve degenerates to the union of a hyperelliptic curve of