$\Theta(Z)$ is a Siegel modular form for Γ_2 of weight ten, and

$$D = \mathrm{div}(\Theta).$$

The key to prove this is to show that $Z \in H_2$ is reducible if and only if one theta constant $\theta\begin{bmatrix} u \\ v \end{bmatrix}$ vanishes at Z. Hammond [15] has proved this by using moduli theory, that is, if $\theta\begin{bmatrix} u \\ v \end{bmatrix}(Z)=0$, then the principally polarized abelian variety corresponding to $Z \in H_2$ is decomposable. Freitag [6, 10] has done in a way similar to what Gundlach [13] did, namely by the function-theoretic argument of theta constants.

D is in the case $D(1)$, and hence $A(\Gamma_2)^{(2)}$ equals $C[E_4, E_6, E_{12}, \Theta]$ by the argument of § 2. Θ is equal to $E_4E_6 - E_{10}$ up to a constant factor. So we have the following:

Theorem (Igusa). *$A(\Gamma_2)^{(2)} = C[E_4, E_6, E_{10}, E_{12}]$, and $E_4, \cdots, E_{12}$ are algebraically independent. The generating function is given by $1/(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})$.*

By this, the Siegel modular function field of degree two is shown to be rational. $A(\Gamma_2)$ is generated by $A(\Gamma_2)^{(2)}$ and a cusp form χ_{35} of weight 35. For this fact and for the definition of χ_{35} we refer the reader to Igusa [30].

Remark. So far, a graded ring of modular forms is written as a finite *free* module over its graded subring isomorphic to a polynomial ring. Such a ring is called Cohen-Macaulay. In the case of real quadratic field K, $A(\Gamma_K)^{(r)}$ is shown to be Cohen-Macualay for any integer $r>1$ ([53]), and moreover it is very likely that $A(\Gamma_K)$ itself is Cohen-Macaulay, which is not generally proved yet. However, unfortunately this nice property does not hold for a general graded ring of modular forms. For instance, neither the graded ring of Hilbert modular forms for K of degree ≥ 3 nor the ring $A(\Gamma_n)$ of Siegel modular forms of degree $n \geq 3$ is Cohen-Macaulay ([53, 54, 58]).

§ 5. Siegel modular forms of degree three

Let $\mathfrak{M}_g$ be the moduli space of smooth curves of genus g. The Torelli map

$$\mathfrak{M}_g \longrightarrow \mathscr{A}_g$$

gives an embedding, and we identify $\mathfrak{M}_g$ with its image. $\mathfrak{M}_g$ is open in $\mathscr{A}_g$ if $g \leq 3$, and is of codimension one if $g=4$. Let $\mathscr{H}_g$ be the locus in $\mathfrak{M}_g$ consisting of hyperelliptic points, and let $\bar{\mathscr{H}}_g$ be its closure in $\mathscr{A}_g$.

A hyperelliptic curve of genus g is given, as an affine curve, by an equation

$$y^2 = \prod_{i=1}^{2g+2} (x - \xi_i)$$

where $\xi_1, \cdots, \xi_{2g+2}$ are mutually distinct complex numbers. Let W_{2g+2} be an open subvariety of C^{2g+2} consisting of points with distinct coordinates, which parametrizes "all" hyperelliptic curves of genus g. There is a surjective morphism of W_{2g+2} onto $\mathscr{H}_g$ by sending $(\xi_1, \cdots, \xi_{2g+2})$ to the isomorphism class of the corresponding hyperelliptic curve. $\mathscr{H}_g$ is a quotient of W_{2g+2} by the composite of $SL_2(C)$ and of the symmetric group $\mathfrak{S}_{2g+2}$ of degree $2g+2$. Let $S(2, 2g+2)$ be the graded ring of invariants of a binary $(2g+2)$-form, which comes out as the invariant subring of $C[\xi_1, \cdots, \xi_{2g+2}]$ under the above-mentioned group where we refer the reader to Schur [42], Tsuyumine [56, Chap. I, II] for detail. $\mathscr{H}_g \to \mathrm{Proj}(S(2, 2g+2))$ is a compactification of $\mathscr{H}_g$. Let $S(2g+2)$ denote the invariant subring of $C[\xi_1, \cdots, \xi_{2g+2}]$ under $SL_2(C)$, so that the invariant subring of $S(2g+2)$ under $\mathfrak{S}_{2g+2}$ equals $S(2, 2g+2)$. If $\Gamma_g(2)$ denotes the principal congruence subgroup of level two, then Igusa [30] has obtained the homomorphism

$$\rho_g\colon A(\Gamma_g(2)) \longrightarrow S(2g+2)$$

of graded rings which is described explicitly in terms of theta constants. ρ_g maps the graded part of degree k to that of degree $gk/2$. He showed also that the restriction of ρ_g to $A(\Gamma_g)$, which we denote also by ρ_g, is a homomorphism of $A(\Gamma_g)$ to $S(2, 2g+2)$;

$$\rho_g\colon A(\Gamma_g) \longrightarrow S(2, 2g+2)$$

which is associated with an embedding $\mathscr{H}_g \longrightarrow \mathscr{A}_g$. The kernel of ρ_g is the ideal generated by modular forms vanishing identically on $\mathscr{H}_g \subset \mathscr{A}_g$. ρ_g is injective if $g \leq 2$ because all curves of genus $g \leq 2$ are hyperelliptic.

Now let us consider the case $g=3$. There are 36 even theta characteristics of degree three. The corresponding product of theta constants

$$\chi_{18} := \prod_{\text{even}} \theta\begin{bmatrix} u \\ v \end{bmatrix}$$

is a cusp form for Γ_3 of weight 18. It was shown by Igusa [30] that