case $D(8)$. It gives one good example which shows how we should treat the case $D(r)$, $r \geq 2$.

§ 4. Siegel modular forms of degree two

Let H_n be the Siegel space of degree n; $\{Z \in M_n(C) | {}^tZ = Z, \operatorname{Im} Z > 0\}$. Let Γ_n be the modular group $\{M \in M_{2n}(Z) | {}^tMJM = J\}$ where J denotes $\begin{bmatrix} 0 & 1_n \\ -1_n & 0 \end{bmatrix}$, 1_n being the identity matrix of size n. Γ_n acts on H_n by the usual modular transformation;

$$Z \longrightarrow MZ = (AZ+B)(CZ+D)^{-1}, \qquad M = \begin{pmatrix} AB \\ CD \end{pmatrix} \in \Gamma_n.$$

The jacobian $j(M, Z)$ at Z, of the automorphism of H_n induced by M, is $|CZ+D|^{-n-1}$. Let $\mathscr{A}_n := H_n/\Gamma_n$, and let $\mathscr{A}_n^*$ be the Satake compactification which is normal and projective. $\mathscr{A}_n$ is the moduli space of principally polarized abelian varieties over C of dimension n. $\operatorname{codim}(\mathscr{A}_n^* - \mathscr{A}_n)$ equals n, and hence the variety $\mathscr{A}_n$ satisfies the first condition in the Assumption I if $n > 1$. Let Γ be a congruence subgroup of Γ_n. A holomorphic function f on H_n is called a *Siegel modular form* for Γ of weight k if it satisfies

$$f(MZ) = |CZ+D|^k f(Z) \qquad \text{for } M = \begin{pmatrix} AB \\ CD \end{pmatrix} \in \Gamma$$

where for $n=1$ we need the additional condition that f is holomorphic also at the cusps, which is automatic in the case $n>1$. We denote by $A(\Gamma)$ the graded ring of Siegel modular froms for Γ. $\mathscr{A}_n^*$ equals $\operatorname{Proj}(A(\Gamma_n))$. We denote by $E_k(Z)$ the Eisenstein series of even weight k, which is an example of Siegel modular forms. We define a theta constant by setting

$$\theta\begin{bmatrix} u \\ v \end{bmatrix}(Z) = \sum_{g \in Z^n} e\left(\frac{1}{2}\left(g+\frac{u}{2}\right)Z\,{}^t\!\left(g+\frac{u}{2}\right)+\frac{1}{2}\left(g+\frac{u}{2}\right){}^t v\right)$$

where u, v as well as g, are integral row vectors of size n. $\begin{pmatrix} u \\ v \end{pmatrix}$ is called a *theta characteristic of degree n*, and is said to be *even* or *odd* according as $e(u\,{}^tv/4) = 1$ or -1. $\theta\begin{bmatrix} u \\ v \end{bmatrix}$ is not identically zero if and only if $\begin{pmatrix} u \\ v \end{pmatrix}$ is even. A point Z of H_n is called *reducible* if it is equivalent to a matrix of the form $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ under Γ_n. A *reducible locus* in $\mathscr{A}_n$ is the subset consisting of all points corresponding to reducible points, which is closed in $\mathscr{A}_n$.

Now let us assume $n=2$. We introduce Freitag [6] (cf. [10]) and Hammond [15] from our point of view, in which Igusa's structure theorem for $A(\Gamma_2)^{(2)}$ was reproved. We take as D the reducible locus of $\mathscr{A}_2$, or equivalently, the divisor corresponding to the image of the embedding;

$$H^2 \longrightarrow H_2.$$
$$(z_1, z_2) \longrightarrow \begin{bmatrix} z_1 & 0 \\ 0 & z_2 \end{bmatrix}$$

The stabilizer subgroup of Γ_2 at the image, is generated by the image of $\Gamma_1^{\times 2}$ by

$$\Gamma_1^{\times 2} \longrightarrow \Gamma_2$$
$$\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right\} \longrightarrow \begin{bmatrix} a & 0 & b & 0 \\ 0 & a' & 0 & b' \\ c & 0 & d & 0 \\ 0 & c' & 0 & d' \end{bmatrix}$$

and by a matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The latter matrix induces the transposition σ of z_1 and z_2. So D is isomorphic to $(H_1/\Gamma_1)^{\times 2}/\langle\sigma\rangle$. Since $(H/\Gamma_1)^* \simeq \operatorname{Proj}(C[g_4, g_6])$, D is associated with a graded ring $B := C[g_4 \otimes g_4, g_6 \otimes g_6, g_4^3 \otimes g_6^2, g_6^2 \otimes g_4^3]^{\langle\sigma\rangle}$ where σ changes the first and second components of a tensor. It is easy to see that

$$B = C[g_4 \otimes g_4, g_6 \otimes g_6, g_4^3 \otimes g_6^2 + g_6^2 \otimes g_4^3].$$

Let

$$\Psi : A(\Gamma_2)^{(2)} \longrightarrow B$$
$$f \longrightarrow f|_{H^2}$$

It is shown that $\Psi(E_4)$, $\Psi(E_6)$, $\Psi(E_{12})$ are algebraically independent. Thus Ψ is surjective. There are ten even theta characteristics of degree two. Let Θ be the square of the corresponding product of theta constants;

$$\Theta(Z) := \prod_{\text{even}} \theta\begin{bmatrix} u \\ v \end{bmatrix}(Z)^2.$$