Then the stabilizer subgroup of $\Gamma_K$ at $H$ is equal to $\Gamma_1 \subset \Gamma_K$. Hence we have an embedding $H/\Gamma_1 \to X_K$. If $f(z)$ is a Hilbert modular form of weight $k$, then $f(\tau, \cdots, \tau)$ is an elliptic modular form of weight $nk$.

Let $K=Q(\sqrt{5})$. We introduce Gundlach [13] from our point of view. Let $D$ be the modular curve in $X_K$ given as above. We have a homomorphism

$$\Psi: A(\Gamma_K) \longrightarrow A(\Gamma_1).$$
$$f \longrightarrow f|_H$$

Let us denote by $G_2(z)$, $G_6(z)$ the Eisenstein series for $\Gamma_K$ of weight 2, 6 respectively. Then it can be shown that $\Psi(G_2(z))$ and $\Psi(G_6(z))$ are algebraically independent. Hence $\Psi|_{A(\Gamma_K)^{(2)}}$ gives rise to a surjective map of $A(\Gamma_K)^{(2)}$ onto $A(\Gamma_1)^{(4)}$. Let $f(z)$ be of odd weight. Then the weight of $\Psi(f)$ is $\not\equiv 0$ (mod 4), and so it is a multiple of $g_6$. If $f$ is a unit with negative norm, then we have $f(\varepsilon z)=-f(z)$. The usual Fourier expansion argument shows that $\Psi(f)$ vanishes at the cusp of order at least two, and hence $\Psi(f)$ is divisible by $\Delta^2$. So $\Psi(f)$ is a multiple of $g_6\Delta^2$ if $f$ is of odd weight. Gundlach [13, p. 246] found out a Hilbert modular form of weight 15, which we denote by $h(z)$, such that $\Psi(h)$ equals $g_6\Delta^2$ up to a constant factor. Now the image of $A(\Gamma_K)\Psi$ by is determined as

$$\Psi(A(\Gamma_K))=A(\Gamma_1)^{(4)}[g_6\Delta^2].$$

We define a theta constant with characteristic $\binom{\alpha}{\beta}$ $\alpha, \beta \in O_K$, by setting

$$\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z)=\sum_{\nu} e\left(\operatorname{tr}\left(\left(\frac{1}{2\sqrt{5}}\left(\nu+\frac{\alpha}{2}\right)^2 z+\frac{1}{2\sqrt{5}}\left(\nu+\frac{\alpha}{2}\right)\beta\right)\right)\right)$$

where $e(\ ):=\exp(2\pi\sqrt{-1}(\ ))$ and where $\nu$ runs over $O_K$, and $\operatorname{tr}(\nu z):=\nu^{(1)}z_1+\cdots+\nu^{(n)}z_n$. A theta characteristic $\binom{\alpha}{\beta}$ is said to be *even* or *odd* according as $e(\operatorname{tr}(\alpha\beta)/4)=1$ or $-1$, and $\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$ is not identically zero if and only if $\binom{\alpha}{\beta}$ is even. There are ten even theta characteristics mod 2, and the corresponding product

$$\Theta_0(z):=\prod_{\text{even}} \theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z)$$

is a moldular form of weight 5. With the aid of Götsky's observation, Gundlach obtained after some calculation, that



$$\operatorname{div}(\Theta_0)=D,$$

which is obtained also by the modular embedding argument as well as the result of Siegel modular forms of degree two (Hammond [16]). So $D$ satisfies the Assumption II and is in the case $D(1)$. By the argument of the preceding section we have the following:

**Theorem** (Gundlach). *Let* $K=Q(\sqrt{5})$. *Then* $A(\Gamma_K)=C[G_2, G_6, \Theta_0, h]$. *The generating function* $P_{A(\Gamma_K)}(t)$ *equals* $(1+t^{15})/(1-t^2)(1-t^5)(1-t^6)$.

Let $K$ be an arbitrary real quadratic field. Let $\sigma$ be the automorphism of $H^2$ given by

$$z=(z_1, z_2) \longrightarrow \sigma z=(z_2, z_1).$$

A Hilbert modular form $f$ is said to be *symmetric* if $f(z_1, z_2)=f(z_2, z_1)$. Let $\hat{\Gamma}_K$ be the composite of $\Gamma_K$ and $\langle\sigma\rangle$ as groups acting on $H^2$, and let $\hat{X}_K=H^2/\hat{\Gamma}_K \simeq X_K/\langle\sigma\rangle$. We denote by $A(\hat{\Gamma}_K)$ the graded ring of symmetric Hilbert modular forms. $\hat{X}_K$ has a natural compactification $\hat{X}_K^*:=X_K^*/\langle\sigma\rangle$ which equals $\operatorname{Proj}(A(\hat{\Gamma}_K))$.

We return to the case $K=Q(\sqrt{5})$. Let $D'$ be the irreducible divisor of $\hat{X}_K$ defined by $z_1=z_2$, in other words, the image of $D$ by the natural surjective map of $X_K$ onto $\hat{X}_K$. In the above-mentioned system of generators of $A(\Gamma_K)$, $G_2$, $G_6$, $h$ are symmetric and $\Theta_0$ is anti-symmetric, i.e., $\Theta_0(z_2, z_1)=-\Theta(z_1, z_2)$. Then $\Theta_0^2$ is symmetric and

$$\operatorname{div}(\Theta_0^2)=D'.$$

So $D'$ is in the case $D(1)$, and we have the following:

**Theorem** (Gundlach). *Let* $K=Q(\sqrt{5})$. *Then* $A(\hat{\Gamma}_K)=C[G_2, G_6, \Theta_0^2, h]$. *The generating function* $P_{A(\hat{\Gamma})}(t)$ *equals* $(1+t^{15})/(1-t^2)(1-t^6)(1-t^{10})$.

The image of $A(\hat{\Gamma}_K)^{(2)}$ by $\Psi$ equals $A(\Gamma_1)^{(4)}=C[g_4, g_6^2]$, and the kernel is the ideal generated by $\Theta_0^2$. By the argument in § 2, $A(\Gamma_K)^{(2)}$ is shown to be equal to $C[G_2, G_6, \Theta_0^2]$ which is isomorphic to a polynomial ring. This shows in particular that the symmetric Hilbert modular function field for $Q(\sqrt{5})$ is rational. $h^2$ is written as a polynomial of $G_2$, $G_6$, $\Theta_0^2$, which has been explicitly done in Resnikoff [36], Hirzebruch [21].

Success in this line depends on finding out a "good" divisor of $X_K$ or $\hat{X}_K$. We refer the reader to Hammond [16], Hermann [17, 18] for the case of real quadratic field $K$ other than $Q(\sqrt{5})$. Although Hermann [18] might look like the case when $D$ is not irreducible, it is actually the