$$
\begin{array}{ccccc}
A(i) & \longrightarrow & A(i+\nu(g)) & \longrightarrow & \bar{A}' \\
f & \longrightarrow & fg & \longrightarrow & (fg/\chi^{(i+\nu(g))/r})|_D .
\end{array}
$$

$\Psi(0)$ equals Ψ by definition. The kernel of $\Psi(i)$ is just $A(i+1)$ and hence $\Psi(i)$ is regarded also as an injective map of $\bar{A}(i)$ into $\bar{A}'$ which is an $\bar{A}$-module homomorphism. By definition, the map $A(i)$ to $\bar{A}'$ given by $f \to \Psi(i+r)(\chi f)$ equals $\Psi(i)$, and in particular there is an inclusion $\Psi(i)(\bar{A}(i)) \subset \Psi(i+r)(\bar{A}(i+r))$. If we identify $\bar{A}(i)$ with its $\Psi(i)$-image, then we have r ascending sequences of $\bar{A}$-modules;

$$
\begin{gathered}
\bar{A}(0) \subset \bar{A}(r) \subset \bar{A}(2r) \subset \cdots \\
\bar{A}(1) \subset \bar{A}(r+1) \subset \bar{A}(2r+1) \subset \cdots \\
\vdots \\
\bar{A}(r-1) \subset \bar{A}(2r-1) \subset \bar{A}(3r-1) \subset \cdots .
\end{gathered}
$$

Since all modules are submodules of $\bar{A}'$ and since $\bar{A}'$ is finite over $\bar{A}$, there is an integer i_0 such that

$$\bar{A}(i) = \bar{A}(i+r) \qquad \text{for any } i \geq i_0,$$

in other words, $\chi\bar{A}(i) = \bar{A}(i+r)$. Then if S is a graded subring of $\bar{A}$ isomorphic to a polynomial ring, then A is isomorphic as graded S-modules, to $\bar{A}(0) \oplus \bar{A}(1) \oplus \cdots \oplus \bar{A}(i_0-1) \oplus \bigoplus_{\mu=0}^{\infty} (\bar{A}(i_0) \oplus \cdots \oplus \bar{A}(i_0+r-1))\chi^{\mu}$, where A is regarded as an S-module by taking some homomorphism of S into A for which $S \to A \xrightarrow{\Psi} \bar{A}$ is the identity on S. Now suppose that the structures of $\bar{A}$ and of a finite number of $\bar{A}$-submodules $\bar{A}(i)$, $1 \leq i \leq i_0 + r - 1$, in $\bar{A}'$ are determined. Then *the generating function of A is given by*

$$\sum_{i=0}^{i_0-1} P_{\bar{A}(i)}(t) + (1 - t^{\deg(\chi)}) \left(\sum_{i=i_0}^{i_0+r-1} P_{\bar{A}(i)}(t) \right).$$

If $\{f_{j,0}\}$ (resp. $\{f_{j,i}\}$, $i \geq 1$) is any system of homogeneous elements in A (resp. $A(i)$) whose images by Ψ (resp. $\Psi(i)$) generate $\bar{A}$ (resp. $\bar{A}(i)$ over $\bar{A}$), then A is generated by χ and $f_{j,i}$'s with $i \leq i_0 + r - 1$. To this assertion, a proof similar to the case $D(1)$ can be given, but we omit it.

Suppose that X, $\mathscr{L}$ satisfy the Assumption I. Let

$$X_0 \subset X_1 \subset \cdots \subset X_m = X$$

be a sequence of inclusions of irreducible subvarieties of X such that the codimensions of X_j in X_{j+1} is one for every $j=0, \cdots, m-1$. Furthermore suppose that the closure $\bar{X}_j$ of X_j in X^* satisfies a condition that the

codimension of $\bar{X}_j - X_j$ in $\bar{X}_j$ is at least two for $j \geq 1$. Then X_j and $\mathscr{L}|_{X_j}$, $j \geq 1$, satisfy the Assumption I. If a divisor X_j in X_{j+1} satisfies the Assumption II for each $j=0, \cdots, m-1$ and if the above trick is successful for each j, then we may have some information about the graded ring associated with X, $\mathscr{L}$ from the graded ring $\oplus_k H^0(X_0^*, \mathscr{L}(k)^*|_{X_0^*})$, X_0^* denoting the closure of X_0 in X^*.

§ 3. Hilbert modular forms

Let K be a totally real algebraic number filed of degree $n > 1$, and let O_K be the maximal order of K. $SL_2(K)$ acts on the product H^n on n copies of the upper half plane $H = \{z_1 \in C \mid \operatorname{Im} z_1 > 0\}$ by the modular substitution;

$$z = (z_1, \cdots, z_n) \longrightarrow Mz = \left(\frac{\alpha^{(1)} z_1 + \beta^{(1)}}{\gamma^{(1)} z_1 + \delta^{(1)}}, \cdots, \frac{\alpha^{(n)} z_n + \beta^{(n)}}{\gamma^{(n)} z_n + \delta^{(n)}} \right)$$

where $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(K)$ and $\alpha^{(1)}, \cdots, \alpha^{(n)}$ denote the conjugates of $\alpha \in K$. The jacobian $j(M, z)$ at z, of the automorphism of H^n induced by M, is $N(\gamma z + \delta)^{-2} = \prod_{i=1}^{n} (\gamma^{(i)} z + \delta^{(i)})^{-2}$. Let Γ_K denote the Hilbert modular group $SL_2(O_K)$. Γ_K acts properly discontinuously on H^n. Let $X_K := H^n/\Gamma_K$. X_K has a natural compactification X_K^*, which is given by adding h points called cusps to X_K, h denoting the class number of K, and X_K^* is normal and projective. A holomorphic function f on H^n is called a *Hilbert modular form* for Γ_K of weight $k \in Z, \geq 0$ if it satisfies

$$f(Mz) = N(\gamma z + \delta)^k f(z), \qquad M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_K .$$

We denote by $A(\Gamma_K) = \oplus_k A(\Gamma_K)_k$ the graded ring of Hilbert modular forms. X_K^* is canonically isomorphic to $\operatorname{Proj}(A(\Gamma_K))$.

For later use, we fix some notations of elliptic modular functions. Let $\Gamma_1 = SL_2(Z)$. $A(\Gamma_1)$ denotes the graded ring of elliptic modular forms for Γ_1. $g_4(\tau)$, $g_6(\tau)$, $\tau \in H$, denote the Eisenstein series of weight 4, 6 respectively, and $\Delta(\tau)$ denotes the cusp form of weight 12 which is written as a polynomial of g_4, g_6. $A(\Gamma_1)$ is generated by g_4, g_6, and $A(\Gamma_1)^{(4)}$ is generated by g_4 and g_6^2.

We embed H diagonally into H^n;

$$
\begin{aligned}
H &\longrightarrow H^n \\
\tau &\longrightarrow (\tau, \cdot \cdot, \tau).
\end{aligned}
$$