$$\Psi : A \longrightarrow B$$
$$f \longrightarrow f|_D$$

which is associated with the inclusion map of D into X, or D^* into X^*. We write

$$\bar{A} := \operatorname{Im} \Psi$$

which is a graded subring of B. Now we make the following assumption on D.

Assumption II. There is a homogeneous element χ in A such that χ vanishes only at D. Any such element is equal to a power of χ up to a constant factor.

Under the first condition in the Assumption II, the existence of χ satisfying the second is equivalent to the assertion that if $\varphi: \tilde{X} \to X$ is the normalization, then $\tilde{D} = \varphi^{-1}(D)$ is irreducible. For a positive rational number r we call $D(r)$ the following condition on D; $D(r)$: χ defines $r\tilde{D}$ in $\tilde{X}$ where χ is regarded as an element of $\oplus_k H^0(\tilde{X}, \varphi^*(\mathscr{L}(k)))$. For simplicity, we shall deal only with the case r integral, because the similar argument is applicable to the general case. We note that $D(1)$ is the case that D is defined ideal-theoretically by χ in X. χ defines also a subscheme of X^*, which equals D^* set-theoretically. All irreducible subvarieties in X of codimension one satisfy $D(r)$, $r \leq r_X$ for some fixed $r_X \in Z$, if $\operatorname{Pic}(X) \simeq Z + \{\text{finite torsion}\}$, which is the case if $X = \mathscr{D}/\Gamma$ and $\mathscr{L}$ are as in the Example 1 where $\mathscr{D}$ is an irreducible bounded symmetric domain under a certain condition and Γ has a finite commutator factor group (cf. Tsuyumine [55]). All of them satisfy in $D(1)$ particular if $\operatorname{Pic}(X) \simeq Z$ and if an automorphy factor ρ is taken suitably (loc. cit., see also Freitag [8, 9]).

The Assumption II implies that the divisor nD, for some positive integer n, corresponding to an ample invertible sheaf, and hence that Ψ is surjective for graded parts of sufficiently large degree which is $\equiv 0$ (mod d'), d' being sufficiently divisible.

Let us consider the primitive case $D(1)$. Suppose that the structure of $\bar{A}$ is known, e.g., B is known and Ψ is surjective which is often the case (but not always) in the study of rings of automorphic forms. Then we can deduce some information on A from $\bar{A}$. *In the case of $D(1)$, the generating function $P_A(t)$ of A is given by $(1-t^{\deg(\chi)})P_{\bar{A}}(t)$, $\deg(\chi)$ denoting the degree of χ in A. If $\{f_i\}$ is any system of homogeneous elements in A whose images by Ψ generate $\bar{A}$, then A is generated by χ and the f_i's.* *In particular if the ring $\bar{A}$ is isomorphic to a polynomial ring, then A is too, and furthermore A and $\bar{A}[\chi]$ are isomorphic as graded rings.* The first assertion follows from an exact sequence

$$0 \longrightarrow (\chi) \longrightarrow A \xrightarrow{\Psi} \bar{A} \longrightarrow 0.$$

Let f be any homogeneous element in A, and let $k = \deg(f)$. We are able to prove by induction on k, that f is contained in the ring generated by χ and the f_i's. If $k<0$, then the assertion is trivial. Suppose that it is true for an element of degree $<k$. $\Psi(f)$ is written as a polynomial $Q(\Psi(f_1), \cdots)$ of $\Psi(f_i)$'s. Then the image of $f - Q(f_1, \cdots)$ by Ψ vanishes, and hence it is divisible by χ. If $f - Q(f_1, \cdots) = \chi g$, then g is of degree $<k$, and by the induction hypothesis g is written by χ and the f_i's, and hence f is. The third assertion is immediate from the fact that in the above exact sequence Ψ has a section by assumption.

The general case is to be treated somewhat delicately. Let $\varphi' : \tilde{X}^* \to X^*$ be the normalization, which is an extension of $\varphi: \tilde{X} \to X$. Let $\tilde{D}^*$ be the closure of $\tilde{D}$ in $\tilde{X}^*$, and let $\bar{A}' := \oplus_k H^0(\tilde{D}^*, \varphi'^*(\mathscr{L}(k)^*)|_{\tilde{D}^*})$, which is a homogeneous coordinate ring of $\tilde{D}^*$. Since $\varphi'|_{\tilde{D}^*}: \tilde{D}^* \to D^*$ is a finite morphism, it is shown that $\bar{A}'$ is finite over $\bar{A}$ as a module. Now let us define a valuation on A in terms of $\tilde{D}$. Let $f \in A$ be a homogeneous element. Then for some integer $m>0$, f^m can be regarded as a global section of the invertible sheaf $\mathscr{L}(k)$, $k \equiv 0$ (mod d), and hence that of the invertible sheaf $\varphi^*(\mathscr{L}(k))$ on $\tilde{X}$. Then $\nu(f^m)$ is defined to be the vanishing order of f^m at $\tilde{D}$, and $\nu(f)$ is defined to be $\nu(f^m)/m$ ($\nu(0) = +\infty$). It is easy to check that ν is well-defined. If $\nu(f) = k'r$ for an integer k', then $f/\chi^{k'}$ defines a global section of $\varphi^*(\mathscr{L}(k))$ for $k = \deg(f) - \deg(\chi^{k'})$ (not necessarily that of $\mathscr{L}(k)$), and so $(f/\chi^{k'})|_{\tilde{D}} \in \bar{A}'$. We denote again by Ψ the map $f/\chi^{k'} \to (f/\chi^{k'})|_{\tilde{D}}$, which is an extension of the previous Ψ. For an integer i, let $A(i)$ be the ideal of A generated by homogeneous elements f with $\nu(f) \geq i$. There is a filtration

$$A = A(0) \supset A(1) \supset A(2) \supset \cdots.$$

Let

$$\bar{A}(i) := A(i)/A(i+1).$$

$\bar{A}(0)$ equals $\bar{A}$, which is a noetherian graded ring, and $\bar{A}(i)$'s are noetherian graded modules over $\bar{A}$. Let us fix some i'. If k' is large enough, then there is a homogeneous element g in $A(k'r - i') - A(k'r - i' + 1)$, where g is taken to be 1 if $i' \equiv 0$ (mod r). Fixing such g, we define a map $\Psi(i)$ of $A(i)$ of $\bar{A}'$ for $i \equiv i'$ (mod r), by