Let $i: X \hookrightarrow X^*$ be the natural injection. We write $\mathscr{L}(k)^* = i_*(\mathscr{L}(k))$, which is a coherent sheaf on X^* and which is an ample invertible sheaf if $k \equiv 0 \pmod{d}$. $H^0(X^*, \mathscr{L}(k)^*)$ is equal to $H^0(X, \mathscr{L}(k))$ for any k. There is a finite morphism of X^* onto $\bar{X}$ which is the identity on X. X^*, $\bar{X}$ are not necessarily equal, however they are when $\bar{X}$ is normal.

Let $A_k = H^0(X, \mathscr{L}(k))$, which is the graded part in A of degree k, so that $A = \oplus_k A$. A formal power series $P_A(t)$ in a variable t is called *the generating function of A* if

$$P_A(t) = \sum_k (\dim_C A_k) t^k.$$

Since A is noetherian, $P_A(t)$ is a rational function of t by the Hilbert theorem.

Let V be an irreducible subvariety in X, and let $\bar{V}$ be the closure of V in X^*. We note that if $\bar{V} - V$ is of codimension at least two in $\bar{V}$, then V and $\mathscr{L}|_V$ satisfy the Assumption I.

Example 1. Let $\mathscr{D}$ be a bounded symmetric domain. $\mathscr{D}$ is written as a quotient $G(\boldsymbol{R})^\circ/K$ where $G(\boldsymbol{R})^\circ$ is the (topological) identity component of the group of real points of a semi-simple algebraic group G over $\boldsymbol{Q}$, and K is a maximal compact subgroup of $G(\boldsymbol{R})^\circ$. Let Γ be an arithmetic subgroup of G, which acts properly discontinuously on $\mathscr{D}$. By Baily-Borel [1], there is a natural compactification $(\mathscr{D}/\Gamma)^*$ which is called a Baily-Borel-Satake compactification. It is a normal projective algebraic variety and has $\mathscr{D}/\Gamma$ as an open subvariety. Now let us suppose that G has no normal $\boldsymbol{Q}$-subgroup of dimension three with $\boldsymbol{Q}$-rank one. Then $\operatorname{codim}((\mathscr{D}/\Gamma)^* - \mathscr{D}/\Gamma)$ is at least two. So if we put $X = \mathscr{D}/\Gamma$, then X satisfies the first condition in the Assumption I. We take as $\mathscr{L}$, the quasi-coherent sheaf corresponding to automorphic forms for Γ on $\mathscr{D}$, whose details are given in the following.

Let ρ be a (holomorphic) automorphy factor, i.e., the function on $\Gamma \times \mathscr{D}$ with values in $\boldsymbol{C} - \{0\}$ such that (i) $\rho(\gamma, z)$ is holomorphic in z for any fixed $\gamma \in \Gamma$, (ii) $\rho(\gamma\gamma', z) = \rho(\gamma, \gamma' z) \cdot \rho(\gamma', z)$ for $\gamma, \gamma' \in \Gamma$, and (iii) $\rho(\gamma, z) = \rho(\gamma', z)$ if γ, γ' induce the same automorphism of $\mathscr{D}$. Let $j(\gamma, z)$ be the jacobian at a point $z \in \mathscr{D}$, of an automorphism of $\mathscr{D}$ induced by $\gamma \in \Gamma$. $j(\gamma, z)$ is an example of automorphy factors. Let us consider the automorphy factor ρ which is of the form

$$\rho(\gamma, z) = v(\gamma) j(\gamma, z)^{-r}$$

where r is a positive rational number and v is a multiplier whose value for $\gamma \in \Gamma$ is an m-th root of unity for some fixed $m \in \boldsymbol{Z}$. v is depending on the choice of the branches of $j(\gamma, z)^{-r}$ if $r \in \boldsymbol{Q} - \boldsymbol{Z}$. Let π denote the canonical projection of $\mathscr{D}$ onto $\mathscr{D}/\Gamma$. We define $\mathscr{L}(k)$ to be the coherent sheaf corresponding to ρ^k-automorhpic forms, i.e., the sheaf defined by

$$H^0(U, \mathscr{L}(k)) = \{f \in \mathcal{O}_{\pi^{-1}(U)} \mid f(\gamma z) = \rho(\gamma, z)^k f(z) \quad \text{for } z \in \pi^{-1}(U), \gamma \in \Gamma\},$$

where U is any analytic open subset of $\mathscr{D}/\Gamma$, and $\mathcal{O}_{\pi^{-1}(U)}$ denotes the structure sheaf of $\pi^{-1}(U)$ in analytic sense. $\mathscr{L}(k)$ extends to a coherent sheaf on $(\mathscr{D}/\Gamma)^*$ (Serre [44]), and in particular it is algebraic. By Baily-Borel [1], $\mathscr{L}(k)$ extends to an ample invertible sheaf on $(\mathscr{D}/\Gamma)^*$ if k is sufficiently divisible.

So $X = \mathscr{D}/\Gamma$ and $\mathscr{L} = \oplus_k \mathscr{L}(k)$ satisfy the conditions in the Assumption I, if G has no normal $\boldsymbol{Q}$-subgroup of dimension three with $\boldsymbol{Q}$-rank one. $A_k = H^0(X, \mathscr{L}(k))$ is the space of ρ^k-automorphic forms, i.e., holomorphic functions f on $\mathscr{D}$ satisfying $f(\gamma z) = \rho(\gamma, z)^k f(z)$ for $\gamma \in \Gamma$. $A = \oplus_k A_k$ is the graded ring of such automorphic forms.

Example 2. Let $\mathscr{D}$, Γ be as above. Let X be an irreducible subvariety of $\mathscr{D}/\Gamma$ such that the closure $\bar{X}$ of X in $(\mathscr{D}/\Gamma)^*$ satisfies that $\operatorname{codim}(\bar{X} - X) \geq 2$. Then X and the sheaf given by restricting to X, the quasi-coherent sheaf on $\mathscr{D}/\Gamma$ corresponding to automorphic forms, satisfy the conditions in the Assumption I. Such is the moduli space $\mathfrak{M}_g$ of curves of genus $g \geq 3$, which we discuss later.

Remark. Let the notation be as in the Example 1. To investigate the structure of A, usually we had better to take $r > 0$ as small at possible. If otherwise, it might make the structure of A unnecessarily complicated. For instance, if we compare a polynomial ring and its subring consisting of polynomials of degree $\equiv 0 \pmod{d}$ for $d > 1$, then the former is easier to handle than the latter.

§ 2. A graded ring and a subring

Let X, $\mathscr{L}$ be as in the preceding section. Let D be an irreducible subvariety in X of codimension one, and let D^* be the closure of D in X^*. Let

$$B := \bigoplus_k B_k, \qquad B_k := H^0(D^*, \mathscr{L}(k)^*|_{D^*}).$$

B is noetherian and is a homogeneous coordinate ring of D^*, hence $D^* = \operatorname{Proj}(B)$. A global section f of $\mathscr{L}(k)$ on X can be regarded as that of $\mathscr{L}(k)^*$ on X^*, and hence $f|_D$ determines the unique element of B_k whose restriction to D equals $f|_D$. We have a homomorphism of graded rings