relying on the formula by Cohn [4], and the work by Satoh [41] which is concerned with the module structure of the space of vector valued Siegel modular forms of degree two, was done by relying on Tsushima's formula [51, 52].

The works of this kind have been done after algebraic geometry was fully developed, by which we never mean that all the important methods and ingredients have come from algebraic geometry. The theory of compactification introduced by Satake [39, 40] (cf. [43]) makes us apply algebraic geometry effectively to arithmetic quotients of the Siegel spaces. The works by Igusa [28, 30] have, more or less, flavor of the moduli theory of hyperelliptic curves and of abelian varieties. In the Hilbert modular case, Hirzebruch [19–24] (see also Hirzebruch-Van de Ven [25], Hirzebruch-Zagier [26, 27]) has developed an elegant and powerful geometric method. However it can be said that although we are now equipped with several excellent tools particularly for low dimensional cases, there is still plenty of difficulty because graded rings of modular forms have a character of their own and each of them requires us to treat it in distinct way.

We do not try in the present survey paper such an awful thing as looking round all the tools and the technique to determine the graded rings which have ever been developed. We introduce the method first employed by Gundlach [13] which, or whose modification, is used repeatedly by several authors (Siegel modular case; Hammond [15], Freitag [6], Tsuyumine [56]: Hilbert modular case; Hammond [16], Hermann [17, 18]). After preliminaries in § 1, we introduce it in § 2. In the remaining sections, its applications are shown. In § 3, Gundlach's original argument is introduced from our point of view. In § 4, the structure theorem for the graded ring of Siegel modular forms of degree two is discussed. It is based on the papers by Freitag and by Hammond. In § 5, Siegel modular forms of degree three is discussed by sketching [56], and in the last section we give a small observation about Siegel modular forms of degree four. We expect that the method will be still useful in more cases, and wish that the present paper would be of benefit to people trying further investigation.

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## § 1. A graded ring

We start with a quasi-projective variety $X$ over $C$ and a quasi-



coherent sheaf $\mathscr{L}$ on $X$ on which the following assumption is made:

**Assumption I.** $X$ has a projective compactification $\bar{X}$ such that $\operatorname{codim}(\bar{X}-X)\geq 2$. $\mathscr{L}$ is a torsion-free graded $\mathcal{O}_X$-module $\mathscr{L}=\oplus_k\mathscr{L}(k)$, $k\in Z$, $\geq 0$, with $\mathscr{L}(0)=\mathcal{O}_X$ where $\mathscr{L}(k)$ is coherent and is locally free of rank one except on proper subvarieties. There is a positive integer $d$ such that if $k>0$ is divisible by $d$, then $\mathscr{L}(k)$ is an ample invertible sheaf on $X$ which has an extension to an ample invertible sheaf on $\bar{X}$.

For $k$, $k'\geq 0$, $\mathscr{L}(k)\otimes\mathscr{L}(k')$ is contained in $\mathscr{L}(k+k')$ as an $\mathcal{O}_X$-submodule. The equality holds if $k$, $k'\equiv 0 \pmod d$. It is true also for extensions of $\mathscr{L}(k)$'s, $k\equiv 0 \pmod d$, for the extended invertible sheaves are unique from our assumption that $\operatorname{codim}(\bar{X}-X)\geq 2$. The ampleness of $\mathscr{L}(d)$ implies in particular that $\mathcal{O}_X\subset\mathscr{L}(k)$ if $k$ is large enough. Let $d_k\equiv 0 \pmod d$ be such that $k\leq d_k$ and $\mathcal{O}_X\subset\mathscr{L}(d_k-k)$. Then for any non-negative integer $m$, we have $\mathscr{L}(k+md)\subset\mathscr{L}(d_k+md)=\mathscr{L}(d_k)\otimes\mathscr{L}(d)^m$. There is an ascending sequence of $\mathcal{O}_X$-submodules of $\mathscr{L}(d_k)$; $\mathscr{L}(k)\subset\mathscr{L}(k+d)\otimes\mathscr{L}(d)^{-1}\subset\mathscr{L}(k+2d)\otimes\mathscr{L}(d)^{-2}\subset\cdots\subset\mathscr{L}(d_k)$. The coherency shows that $\mathscr{L}(k+md)\otimes\mathscr{L}(d)^{-m}$ are all equal for large $m$. We have shown that $\mathscr{L}(k)\otimes\mathscr{L}(k')=\mathscr{L}(k+k')$ if $k'\geq 0$ is $\equiv 0 \pmod d$ and if $k$ is large enough, or $\equiv 0 \pmod d$. Hence $\mathscr{L}$ is finite over the submodule $\oplus_k\mathscr{L}(k)$, $k\equiv 0 \pmod d$.

By assumption $H^0(X, \mathscr{L}(k))$ is finite dimensional for any $k\geq 0$. Since by a standard argument (see, for instance, Mumford [34]) $H^0(X, \mathscr{L}(k))\otimes H^0(X, \mathscr{L}(k'))\to H^0(X, \mathscr{L}(k+k'))$ is surjective for sufficiently large $k$, $k'$ which are $\equiv 0 \pmod d$, a graded ring

$$A^{(d)}:=\bigoplus_k H^0(X, \mathscr{L}(k))$$

is noetherian. Let

$$A:=\bigoplus_k H^0(X, \mathscr{L}(k)),$$

which we call *the graded ring associated with $X$ and $\mathscr{L}$*, or *the graded ring associated with $X$* when the quasi-coherent sheaf under consideration is obvious. Then by the above argument the graded ring $A$ is finite over $A^{(d)}$ as a module, and hence noetherian. Let

$$X^*:=\operatorname{Proj}(A).$$

$X^*$ is a compactification of $X$ satisfying the condition in the Assumption I, namely it is a projective variety satisfying

$$\operatorname{codim}(X^*-X)\geq 2.$$