3.2 The case of discriminant 5

Gundlach[5] determined the ring structure of $A_{*,0}$. There are three algebraically independent generators

$$\begin{aligned} G_2 &\in A_{2,0} && (D_0(G_2)=e_4)\,,\\ G_5 &\in A_{5,0} && (D_0(G_5)=0,\ D_1(G_5)=\Delta)\,,\\ G_6 &\in A_{6,0} && (D_0(G_6)=\Delta) \end{aligned}$$

and one algebraically dependent generators

$$G_{15}\in A_{15,0} \qquad \left(D_0(G_{15})=\Delta^2 e_6\right).$$

We remark that G_2, G_6, G_{15} are symmetric and that G_5 is skew-symmetric. Let

$$R := \mathbb{C}[G_2, G_5, G_6].$$

Gundlach showed

$$A_{*,0} = R \oplus RG_{15}.$$

Define

$$\begin{aligned} G_{8,1} &:= [G_2, G_5]_1 \in A_{8,1} && (D_0(G_{8,1})=\Delta e_4)\,,\\ G_{9,1} &:= [G_2, G_6]_1 \in A_{9,1} && (D_0(G_{9,1})=\Delta e_6)\,,\\ G_{12,1} &:= [G_5, G_6]_1 \in A_{12,1} && \left(D_0(G_{12,1})=-3\Delta^2\right),\\ G_{6,2} &:= [G_2, G_2]_2 \in A_{6,2} && (D_0(G_{6,2})=864\Delta) \end{aligned}$$

and

$$G_{9,2} := [G_2, G_5]_2 \in A_{9,2} \qquad (D_0(G_{9,2})=3\Delta e_6).$$

Now we have proved the following theorem:

Theorem 8. *$A_{*,2}$ is a free R-module generated by $G_{6,2}$ and $G_{9,2}$. Namely,*

$$A_{*,2} = RG_{6,2} \oplus RG_{9,2}.$$

Lemma 9. *We have three equations:*
(1) $\dim_{\mathbb{C}} A_{5n+12,1}(n) = 2$. $A_{5n+12,1}(n) = \mathbb{C}G_5^n G_2^2 G_{8,1} \oplus \mathbb{C}G_5^n G_{12,1}$.
(2) $\dim_{\mathbb{C}} A_{5n+8,1}(n) = 1$. $A_{5n+8,1}(n) = \mathbb{C}G_5^n G_{8,1}$.
(3) $A_{5n+6,1}(n) = \{0\}$.

Proof. From Lemma 4, easily we have (1)(2) and $\dim_{\mathbb{C}} A_{5n+6,1}(n) \le 1$. Assume that there exist $F \in A_{5n+6,1}(n)$ such that $D_n(F) = \Delta^{n+1}$. Because $D_n(G_5^n G_2^2 G_{8,1}) = \Delta^{n+1} e_4^3$, $D_n(G_5^n G_{12,1}) = -3\Delta^{n+2}$ and $D_n(G_5^n G_{8,1}) = \Delta^{n+1} e_4$, we have $G_2 F = G_5^n G_{8,1}$ and $-3G_6 F = G_5^n G_{12,1}$. Hence we have

$$0 = 6G_6 G_5^n G_{8,1} + 2G_2 G_5^n G_{12,1} = 5G_5^{n+1} G_{9,1}.$$

This is a contradiction. □

Theorem 10. *$A_{*,1}$ is generated by $G_{8,1}$, $G_{9,1}$ and $G_{12,1}$ as a R-module. The Jacobi identity is a unique relation between these generators. Namely,*

$$A_{*,1} = \mathbb{C}[G_2, G_5]G_{8,1} \oplus RG_{9,1} \oplus RG_{12,1}.$$