3.1 The case of discriminant 12

Gundlach[6] determined the ring structure of $A_{*,0}$. There are three algebraically independent generators

$$\begin{aligned} &G_2 \in A_{2,0} && (D_0(G_2) = e_4)\,, \\ &G_3 \in A_{3,0} && (D_0(G_3) = e_6)\,, \\ &G_4 \in A_{4,0} && (D_0(G_4) = 0,\ D_1(G_4) = 0,\ D_2(G_4) = \Delta) \end{aligned}$$

and one algebraically dependent generators

$$G_{11} \in A_{11,0} \qquad (D_0(G_{11}) = 0,\ D_1(G_{11}) = \Delta^2)\,.$$

We remark that G_2, G_3, G_4 are symmetric (i.e. $F(\tau_1, \tau_2) = F(\tau_2, \tau_1)$) and that G_{11} are skew-symmetric (i.e. $F(\tau_1, \tau_2) = -F(\tau_2, \tau_1)$). Let

$$R := \mathbb{C}[G_2, G_3, G_4].$$

Gundlach showed

$$A_{*,0} = R \oplus RG_{11}.$$

Define

$$\begin{aligned} &G_{6,1} := [G_2, G_3]_1 \in A_{6,1} && (D_0(G_{6,1}) = -864\Delta)\,, \\ &G_{7,1} := [G_2, G_4]_1 \in A_{7,1} && (D_0(G_{7,1}) = 0,\ D_1(G_{7,1}) = \Delta e_4) \end{aligned}$$

and

$$G_{8,1} := [G_3, G_4]_1 \in A_{8,1} \qquad (D_0(G_{8,1}) = 0,\ D_1(G_{8,1}) = \tfrac{3}{2}\Delta e_6)\,.$$

Lemma 6. *We have three equations:*
(1) $\dim_{\mathbb{C}} A_{4n+7,1}(2n+1) = 1$. $A_{4n+7,1}(2n+1) = \mathbb{C}G_4^n G_{7,1}$.
(2) $\dim_{\mathbb{C}} A_{4n+8,1}(2n+1) = 1$. $A_{4n+8,1}(2n+1) = \mathbb{C}G_4^n G_{8,1}$.
(3) $A_{4n+5,1}(2n+1) = \{0\}$.

Proof. From Lemma 2, easily we have (1)(2) and $\dim_{\mathbb{C}} A_{4n+5,1}(2n+1) \leq 1$. Assume that there exist $F \in A_{4n+5,1}(2n+1)$ such that $D_{2n+1}(F) = \Delta^{n+1}$. Because $D_{2n+1}(G_4^n G_{7,1}) = \Delta^{n+1} e_4$ and $D_{2n+1}(G_4^n G_{8,1}) = \frac{3}{2}\Delta^{n+1} e_6$, we have $G_2 F = G_4^n G_{7,1}$ and $\frac{3}{2} G_3 F = G_4^n G_{8,1}$. Hence we have

$$0 = 3G_3 G_4^n G_{7,1} - 2G_2 G_4^n G_{8,1} = 4G_4^{n+1} G_{6,1}.$$

This is a contradiction. □

Theorem 7. *$A_{*,1}$ is generated by $G_{6,1}$, $G_{7,1}$ and $G_{8,1}$ as a R-module. The Jacobi identity is a unique relation between these generators. Namely,*

$$A_{*,1} = \mathbb{C}[G_2, G_3] G_{6,1} \oplus RG_{7,1} \oplus RG_{8,1}.$$

Proof. From Lemma 2 and Lemma 6 (3), The dimension of the $A_{k,1}$ is not greater than the coefficient of x^k on the formal power series development.

$$\frac{x^5 + x^6}{(1-x^2)(1-x^3)(1-x^4)} - \sum_{n=0}^{\infty} x^{4n+5} = \frac{x^6 + x^7 + x^8 - x^{10}}{(1-x^2)(1-x^3)(1-x^4)}.$$

□