Proof. We show this lemma by induction on n. If $n = 0$, $c(0) = 0$ because $c(0) = \varepsilon^l c(0)$ and $l \neq 0$. Now let $F \in A_{k,l}(2n)$. By the assumption of the induction, $c(u, v) = 0$ for any $u < n$. If $v > n$, then

$$c(n, v) = \varepsilon^l c\left(\varepsilon^{-1}\left(\frac{1}{2}n + \frac{\sqrt{3}}{6}v\right)\right) = \varepsilon^l c(2n - v, -3n + 2v) = 0$$

because $2n - v < n$. If $v < -n$, then

$$c(n, v) = \varepsilon^{-l} c\left(\varepsilon\left(\frac{1}{2}n + \frac{\sqrt{3}}{6}v\right)\right) = \varepsilon^{-l} c(2n + v, 3n + 2v) = 0$$

because $2n + v < n$. Hence $F \in A_{k,l}(2n)$ means

$$\sum_{v=-n}^{n} v^r c(n, v) = 0 \quad (r = 0, 1, \ldots, 2n - 1).$$

Easily we have $c(n, n) = \varepsilon^l c(n, -n)$, hence

$$\begin{pmatrix} 1 & 0 & \ldots & 0 & -\varepsilon^l \\ 1 & 1 & \ldots & 1 & 1 \\ n & n-1 & \ldots & -n+1 & -n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ n^{2n-2} & (n-1)^{2n-2} & \ldots & (-n+1)^{2n-2} & (-n)^{2n-2} \\ n^{2n-1} & (n-1)^{2n-1} & \ldots & (-n+1)^{2n-1} & (-n)^{2n-1} \end{pmatrix} \begin{pmatrix} c(n, n) \\ c(n, n-1) \\ c(n, n-2) \\ \vdots \\ c(n, -n+1) \\ c(n, -n) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}.$$

By the Vandermonde formula, we have $c(n, v) = 0$. □

Proposition 3. *The dimension of the $A_{k,l}$ is not greater than the coefficient of x^k on the formal power series development.*

$$\frac{x^5 + x^6}{(1 - x^2)(1 - x^3)(1 - x^4)}$$

Proof. From the previous lemma, the dimension of $A_{k,l}$ is not greater than

$$\begin{aligned} &\sum_{n=0}^{\infty} \dim M_{2k+4n}(n+1) + \sum_{n=0}^{\infty} \dim M_{2k+4n+2}(n+1) \\ =&\sum_{n=0}^{\infty} \dim M_{2(k-4n-6)} + \sum_{n=0}^{\infty} \dim M_{2(k-4n-5)}. \end{aligned}$$

This is the coefficient of x^k on the formal power series development of

$$\sum_{n=0}^{\infty} \frac{x^{4n+6} + x^{4n+5}}{(1 - x^2)(1 - x^3)} = \frac{x^5 + x^6}{(1 - x^2)(1 - x^3)(1 - x^4)}.$$

□