2.2 Fourier coefficients of Hilbert modular forms

Now we investigate these exact sequences more precisely. Assume $F \in A_{k,l}$. Let $c(\nu)$ be the Fourier coefficients of F defined by

$$F(\tau_1, \tau_2) = \sum_{\substack{\nu \in \mathcal{O}^* \\ \nu \geq 0,\ \nu' \geq 0}} c(\nu)\, \mathbf{e}(\nu\tau_1 + \nu'\tau_2).$$

Then easily we have

$$(D_r F)(\tau) = \sum_{\substack{\nu \in \mathcal{O}^* \\ \nu \geq 0,\ \nu' \geq 0}} (\nu - \nu')^r c(\nu)\, \mathbf{e}(\mathrm{tr}(\nu)\tau).$$

From the transformation formula of $F \in A_{k,l}$, we have

$$\begin{array}{ll} c(\varepsilon\nu) = \varepsilon^l c(\nu) & (\varepsilon\varepsilon' = 1) \\ c(\varepsilon^2\nu) = (-1)^{k+l}\varepsilon^{2l} c(\nu) & (\varepsilon\varepsilon' = -1) \end{array}.$$

From now on, we assume $l \neq 0$.

2.2.1 The case of discriminant 12

First, we consider the case $d_K = 12$. Namely, we set

$$K = \mathbb{Q}(\sqrt{3}),\ \mathcal{O} = \mathbb{Z}[\sqrt{3}] \ \text{ and } \ \varepsilon = 2 + \sqrt{3}$$

and

$$\mathcal{O}^* = \left\{ \frac{1}{2}u + \frac{\sqrt{3}}{6}v \ \middle|\ u, v \in \mathbb{Z} \right\}.$$

We remark that $\varepsilon\varepsilon' = 1$. For the sake of simplicity, we put

$$c(u,v) := c\left(\frac{1}{2}u + \frac{\sqrt{3}}{6}v\right).$$

Put

$$\Lambda := \left\{ (u,v) \in \mathbb{Z}^2 \mid |v| \leqq \sqrt{3}u \right\}.$$

Then F has a Fourier expansion

$$F(\tau_1, \tau_2) = \sum_{(u,v) \in \Lambda} c(u,v)\, \mathbf{e}\left(\left(\frac{1}{2}u + \frac{\sqrt{3}}{6}v\right)\tau_1 + \left(\frac{1}{2}u - \frac{\sqrt{3}}{6}v\right)\tau_2 \right)$$

and we have

$$(D_r F)(\tau) = \left(\frac{1}{\sqrt{3}}\right)^r \sum_{(u,v) \in \Lambda} v^r c(u,v)\, \mathbf{e}(u\tau).$$

Lemma 2. *For $n \in \mathbb{N}_0$, if $F \in A_{k,l}(2n)$, then $c(u,v) = 0$ for any $u \leq n$. Hence there exist two exact sequences*

$$0 \longrightarrow A_{k,l}(2n+1) \longrightarrow A_{k,l}(2n) \xrightarrow{D_{2n}} M_{2k+4n}(n+1)$$

and

$$0 \longrightarrow A_{k,l}(2n+2) \longrightarrow A_{k,l}(2n+1) \xrightarrow{D_{2n+1}} M_{2k+4n+2}(n+1).$$