In this exposition, we set the discrete subgroup Γ by

$$\Gamma := \begin{cases} \left\langle \mathrm{SL}_2(\mathcal{O}), \begin{pmatrix} \sqrt{\varepsilon} & 0 \\ 0 & \sqrt{\varepsilon'} \end{pmatrix} \right\rangle & (\varepsilon\varepsilon' = 1) \\ \mathrm{SL}_2(\mathcal{O}) & (\varepsilon\varepsilon' = -1) \end{cases}.$$

For $k_1, k_2 \in \mathbb{Z}$, we say that a holomorphic function F on $\mathbb{H}^2$ is a Hilbert modular form of weight (k_1, k_2) if

$$F(\tau_1, \tau_2) = (c\tau_1 + d)^{-k_1} (c'\tau_2 + d')^{-k_2} F\left(\frac{a\tau_1 + b}{c\tau_1 + d}, \frac{a'\tau_2 + b'}{c'\tau_2 + d'}\right)$$

for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. Because $-\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$, there is no nontrivial Hilbert modular form of weight (k_1, k_2) if $k_1 + k_2$ is odd. Hence we may assume that $k_1 + k_2$ is even. Without loss of the generality, we may assume $k_1 \geq k_2$. Put

$$k := \frac{k_1 + k_2}{2} \in \mathbb{Z}, \qquad l := \frac{k_1 - k_2}{2} \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}.$$

We denote the $\mathbb{C}$-vector space of all Hilbert modular forms of weight (k_1, k_2) satisfying the above condition by $A_{k,l}$. From Koecher principle, $F \in A_{k,l}$ has a Fourier expansion

$$F(\tau_1, \tau_2) = \sum_{\substack{\nu \in \mathcal{O}^* \\ \nu \geq 0,\ \nu' \geq 0}} c(\nu)\, \mathbf{e}(\nu\tau_1 + \nu'\tau_2),$$

where we put $\mathbf{e}(w) = \exp(2\pi\sqrt{-1}w)$. Let

$$A_{*,l} := \bigoplus_{k \in \mathbb{Z}} A_{k,l}.$$

Our purpose is to determine the structure of $A_{*,l}$ as a graded $A_{*,0}$-module.

1.2 Elliptic modular forms

In this section, we quote some notations from [1]. For $k \in \mathbb{Z}$, we say that a holomorphic function f on $\mathbb{H}$ is an elliptic modular form of weight k if

$$f(\tau) = (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right)$$

for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and f has a Fourier expansion

$$f(\tau) = \sum_{n=0}^{\infty} a_f(n)\, \mathbf{e}(n\tau).$$

We denote the $\mathbb{C}$-vector space of all elliptic modular forms of weight k by M_k. Put

$$M_k(n) := \left\{ f \in M_k \;\middle|\; a_f(r) = 0 \text{ if } r < n \right\}.$$

It is well-known that there are two algebraically independent elliptic modular forms

$$e_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)\mathbf{e}(n\tau) \in M_4$$