Corollary 9. *The image by D_r has the following properties.*
(1) If k is even, $D_{2r}(A_k(\Gamma; 2r)) \subset \mathrm{M}^{\mathrm{sym}}_{k+2r}(\widetilde{\Gamma}; r)$.
(2) If k is odd, $D_{2r+1}(A_k(\Gamma; 2r+1)) \subset \mathrm{M}^{\mathrm{skew}}_{k+2r+1}(\widetilde{\Gamma}; r+2)$.

Corollary 10. *There exist two exact sequences.*
(1) If k is even, $A_k(\Gamma) = A_k(\Gamma; 0)$ and

$$0 \longrightarrow A_k(\Gamma; 2r+2) \longrightarrow A_k(\Gamma; 2r) \xrightarrow{D_{2r}} \mathrm{M}^{\mathrm{sym}}_{k+2r}(\widetilde{\Gamma}; r).$$

(2) If k is odd, $A_k(\Gamma) = A_k(\Gamma; 1)$ and

$$0 \longrightarrow A_k(\Gamma; 2r+3) \longrightarrow A_k(\Gamma; 2r+1) \xrightarrow{D_{2r+1}} \mathrm{M}^{\mathrm{skew}}_{k+2r+1}(\widetilde{\Gamma}; r+2).$$

Corollary 11. *We have the upper bounds for the dimensions of $A_k(\Gamma)$.*
(1) If k is even, $\dim_{\mathbb{C}} A_k(\Gamma) \leq \sum_{r=0}^{\infty} \dim_{\mathbb{C}} \mathrm{M}^{\mathrm{sym}}_{k+2r}(\widetilde{\Gamma}; r)$.
(2) If k is odd, $\dim_{\mathbb{C}} A_k(\Gamma) \leq \sum_{r=0}^{\infty} \dim_{\mathbb{C}} \mathrm{M}^{\mathrm{sym}}_{k+2r+1}(\widetilde{\Gamma}; r+2)$.

Now we calculate the Poincaré series of this upper bound. If k is even, we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\sum_{r=0}^{\infty}\left(\dim_{\mathbb{C}} \mathrm{M}^{\mathrm{sym}}_{k+2r}(\widetilde{\Gamma}; r)\right) x^k &= \sum_{r=0}^{\infty} \frac{x^{12r-2r}}{(1-x^4)(1-x^6)(1-x^{12})} \\ &= \frac{1}{(1-x^4)(1-x^6)(1-x^{10})(1-x^{12})}.\end{aligned}$$

If k is odd, we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\sum_{r=0}^{\infty}\left(\dim_{\mathbb{C}} \mathrm{M}^{\mathrm{skew}}_{k+2r+1}(\widetilde{\Gamma}; r+2)\right) x^k &= \sum_{r=0}^{\infty} \frac{x^{12(r+2+1)-(2r+1)}}{(1-x^4)(1-x^6)(1-x^{12})} \\ &= \frac{x^{35}}{(1-x^4)(1-x^6)(1-x^{10})(1-x^{12})}.\end{aligned}$$

Hence, if we construct algebraically independent modular forms of weight $4, 6, 10, 12$, and if we construct a modular forms of weight 35, we finish the proof of Theorem 1 for $N = 1$. Indeed, Igusa [Ig1, Ig2] constructed these modular forms from the theta functions.

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